

Stability and local minimality of spherical harmonic twists $u=Q(|x|)x|x|-1$, positivity of second variations and conjugate points on $SO(n)$

Article (Published Version)

Day, Stuart and Taheri, Ali (2020) Stability and local minimality of spherical harmonic twists $u=Q(|x|)x|x|-1$, positivity of second variations and conjugate points on $SO(n)$. *The Journal of Analysis*, 28 (2). pp. 431-460. ISSN 0971-3611

This version is available from Sussex Research Online: <http://sro.sussex.ac.uk/id/eprint/83619/>

This document is made available in accordance with publisher policies and may differ from the published version or from the version of record. If you wish to cite this item you are advised to consult the publisher's version. Please see the URL above for details on accessing the published version.

Copyright and reuse:

Sussex Research Online is a digital repository of the research output of the University.

Copyright and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable, the material made available in SRO has been checked for eligibility before being made available.

Copies of full text items generally can be reproduced, displayed or performed and given to third parties in any format or medium for personal research or study, educational, or not-for-profit purposes without prior permission or charge, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.



Stability and local minimality of spherical harmonic twists $u = Q(|x|)x|x|^{-1}$, positivity of second variations and conjugate points on $SO(n)$

Stuart Day¹ · Ali Taheri¹

Received: 8 March 2018 / Accepted: 8 May 2019 / Published online: 6 June 2019
© The Author(s) 2019

Abstract

In this paper we discuss the stability and local minimising properties of spherical twists that arise as solutions to the harmonic map equation

$$\mathbf{HME}[u; \mathbb{X}^n, \mathbb{S}^{n-1}] := \begin{cases} \Delta u + |\nabla u|^2 u = 0 & \text{in } \mathbb{X}^n, \\ |u| = 1 & \text{in } \mathbb{X}^n, \\ u = \varphi & \text{on } \partial \mathbb{X}^n, \end{cases}$$

by way of examining the positivity of the second variation of the associated Dirichlet energy. Here, following [31], by a spherical twist we mean a map $u \in \mathcal{W}^{1,2}(\mathbb{X}^n, \mathbb{S}^{n-1})$ of the form $x \mapsto Q(|x|)x|x|^{-1}$ where $Q = Q(r)$ lies in $\mathcal{C}([a, b], \mathbf{SO}(n))$ and $\mathbb{X}^n = \{x \in \mathbb{R}^n : a < |x| < b\}$ ($n \geq 2$). It is shown that subject to a *structural* condition on the twist path the energy at the associated spherical twist solution to the system has a positive definite second variation and subsequently proven to furnish a strong local energy minimiser. A detailed study of Jacobi fields and conjugate points along the twist path $Q(r) = \exp(\mathcal{H}(r)\mathbf{H})$ and geodesics on $\mathbf{SO}(n)$ is undertaken and its remarkable implication and interplay on the minimality of spherical harmonic twists exploited.

Keywords Harmonic maps · Strong local minimisers · Positive second variations · Conjugate points · Compact Lie groups · Weierstrass excess function

Mathematics Subject Classification 35J47 · 35J50 · 53C43 · 49K30 · 22C05 · 53Z05

✉ Ali Taheri
a.taheri@sussex.ac.uk

¹ Department of Mathematics, University of Sussex, Falmer, Brighton, England BN1 9RF, UK

1 Introduction

In this paper we study the stability and local minimising properties of a class of geometrically motivated maps serving as *classical* solutions to the nonlinear harmonic map Eq. [(17) below]. Indeed to describe the set up and tasks more clearly consider the variational energy integral

$$\mathbb{F}[u; \mathbb{X}^n] := \int_{\mathbb{X}^n} F(|\nabla u|^2) dx, \quad (1)$$

where $F = F(t)$ is a non-negative, convex and monotone increasing function of class \mathcal{C}^2 on the half-line $t \geq 0$ having a polynomial type growth at infinity while \mathbb{X}^n is a finite, symmetric, open n -annulus that for definiteness is hereafter taken to be $\mathbb{X}^n = \{x \in \mathbb{R}^n : a < |x| < b\}$ with $n \geq 2$ and $0 < a < b < \infty$.

The admissible map u is assumed to lie in the Sobolev space of sphere-valued maps $\mathcal{S}^p = \mathcal{S}^p(\mathbb{X}) := \{u \in \mathcal{W}^{1,p}(\mathbb{X}^n, \mathbb{S}^{n-1}) : u = \varphi \text{ on } \partial\mathbb{X}^n\}$ where $1 \leq p < \infty$, $\varphi \in \mathcal{C}^1(\partial\mathbb{X}^n, \mathbb{S}^{n-1})$ is a fixed prescribed boundary map and as customary we have written $\mathcal{W}^{1,p}(\mathbb{X}^n, \mathbb{S}^{n-1}) := \{u \in \mathcal{W}^{1,p}(\mathbb{X}^n, \mathbb{R}^n) : |u| = 1 \text{ a.e. in } \mathbb{X}^n\}$. Here and below ∇u denotes the gradient of u and $|\nabla u|$ stands for the Hilbert–Schmidt norm of ∇u , that is, $|\nabla u|^2 = \text{tr}\{[\nabla u]'[\nabla u]\}$.

Now consideration of the vanishing of the first variation of \mathbb{F} at a \mathcal{C}^1 extremal (or equivalently stationary map) u results in the Euler–Lagrange equation, here, the non-linear system

$$\begin{cases} \mathcal{L}[u] = \text{div} [F'(|\nabla u|^2)\nabla u] + F'(|\nabla u|^2)|\nabla u|^2 u = 0 & \text{in } \mathbb{X}^n, \\ |u| = 1 & \text{in } \mathbb{X}^n, \\ u = \varphi & \text{on } \partial\mathbb{X}^n, \end{cases} \quad (2)$$

where F' denotes the derivative of $F = F(t)$ with respect to t and the divergence operator in the first line is understood to act row-wise.¹

One motivation for studying such problems comes from liquid crystals theory and in particular the well-known Oseen–Frank model where the aim is to describe and classify the director fields u arising as extremisers and minimisers of the energy functional

$$\mathbb{F}_{OF}[u; \Omega] = \int_{\Omega} \mathcal{W}(x, u, \nabla u) dx, \quad u \in W^{1,2}(\Omega, \mathbb{S}^{n-1}), \quad (3)$$

subject to suitable boundary conditions. Here $\Omega \subset \mathbb{R}^3$ is a bounded domain representing the body, u is a unit vector-field on Ω (the director field) with ∇u denoting its gradient and the energy density $\mathcal{W} = \mathcal{W}(x, u, \nabla u)$ is given by

¹ A particular solution to this system is the radial projection $u(x) = x|x|^{-1}$ with the choice $\varphi = x|x|^{-1}$ on $\partial\mathbb{X}^n$. For reasons that will be clear later we call this the *trivial* solution. Indeed here $\mathcal{L}[u] = \nabla u \nabla F'(|\nabla u|^2) + F'(|\nabla u|^2)[\Delta u + |\nabla u|^2 u] = 0$ as a result of $F'(|\nabla u|^2)$ depending only on r , $\nabla u = (\mathbf{I}_n - \theta \otimes \theta)/r$ and $\Delta u + |\nabla u|^2 u = 0$.

$$\begin{aligned}\mathcal{W}(x, u, \nabla u) &= k_1 |\nabla \cdot u|^2 + k_2 |u \cdot (\nabla \times u)|^2 + k_3 |u \times (\nabla \times u)|^2 \\ &\quad + (k_2 + k_4)(\operatorname{tr}(\nabla u)^2 - (\nabla \cdot u)^2).\end{aligned}\quad (4)$$

The k_j ($1 \leq j \leq 4$) are the Frank constants that are assumed to satisfy the strict form of Ericksen inequalities [15]: $k_1, k_2, k_3 > 0$, $k_2 > |k_4|$ and $2k_1 > k_2 + k_4$, which result in the coercivity inequality $\mathcal{W}(x, u, \nabla u) \geq \alpha |\nabla u|^2$ for all vector fields u and some $\alpha > 0$. Using the identity

$$\operatorname{tr}(\nabla u)^2 + |u \cdot (\nabla \times u)|^2 + |u \times (\nabla \times u)|^2 = |\nabla u|^2, \quad (5)$$

it is seen that in the case of “equal elastic constants”, that is, $k_1 = k_2 = k_3 =: k$, ($k_4 = 0$) the Oseen-Frank energy reduces to a constant multiple of the Dirichlet energy, here, \mathbb{F} with $F(t) = kt$. Notice also that the term $\operatorname{tr}(\nabla u)^2 - (\nabla \cdot u)^2$ is a null-Lagrangian, that is, its integral depends only on the boundary values of the map u (cf., e.g., [1, 15, 16, 20, 33]). Other motivations come from geometry and calculus of variations; see [13, 14, 16] and the huge list of references therein.

Calculating the second variation of the energy at the extremal u , upon taking a compactly supported smooth $\phi \in \mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$, we obtain the quadratic form

$$\begin{aligned}\frac{d^2}{d\varepsilon^2} \mathbb{F}[u_\varepsilon; \mathbb{X}^n] \Big|_{\varepsilon=0} &= \int_{\mathbb{X}^n} \left(4F''(|\nabla u|^2) \langle \nabla u, \nabla \hat{\phi} \rangle^2 + 2F'(|\nabla u|^2) |\nabla \hat{\phi}|^2 \right. \\ &\quad \left. - 2F'(|\nabla u|^2) |\nabla u|^2 |\hat{\phi}|^2 \right) dx,\end{aligned}\quad (6)$$

where $\hat{\phi} = (\mathbf{I} - u \otimes u)\phi = \phi - \langle \phi, u \rangle u$ is the tangential part of ϕ in that $\langle u, \hat{\phi} \rangle = 0$ everywhere while for $\varepsilon \in \mathbb{R}$ sufficiently small u_ε denotes the radial projection of the unconstrained variation $u + \varepsilon \phi$, that is,

$$u_\varepsilon = \Pi(u + \varepsilon \phi) = \frac{u + \varepsilon \phi}{|u + \varepsilon \phi|}. \quad (7)$$

(See Appendix 1 for the details.) An extremal u of class \mathcal{C}^1 is said to be stable *iff* the above second variation is positive for all non-zero $\phi \in \mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$. Note that by the convexity and monotonicity assumptions on F (i.e., $F' \geq 0$, $F'' \geq 0$), from (6) it follows that

$$\frac{d^2}{d\varepsilon^2} \mathbb{F}[u_\varepsilon; \mathbb{X}^n] \Big|_{\varepsilon=0} \geq 2 \int_{\mathbb{X}^n} F'(|\nabla u|^2) \left\{ |\nabla \hat{\phi}|^2 - |\nabla u|^2 |\hat{\phi}|^2 \right\} dx, \quad (8)$$

with the term inside the curly brackets on the right containing the expression corresponding to the second variation of the Dirichlet energy, that is, (1) with $F(t) \equiv t$. This observation signifies and prompts the study and understanding of the positivity of second variations and stability of extremals in the particular case of the Dirichlet energy as a basis for more general energies with a nonlinear F . Now recall that a spherical twist is a map $u \in \mathcal{C}(\overline{\mathbb{X}}^n, \mathbb{S}^{n-1})$ of the form

$$u : x \mapsto u(x) = \mathbf{Q}(r)\theta = \mathbf{Q}(|x|)x|x|^{-1}, \quad x \in \overline{\mathbb{X}}^n, \quad (9)$$

where $a \leq r = |x| \leq b$ and \mathbf{Q} lies in $\mathcal{C}([a, b], \mathbf{SO}(n))$. The curve $r \mapsto \mathbf{Q}(r)$ is called the *twist path* associated with the spherical twist u and in the event $\mathbf{Q}(a) = \mathbf{Q}(b)$ this closed curve on $\mathbf{SO}(n)$ is called the twist loop associated with u . For spherical twists to be admitted as competing maps in \mathcal{A}_φ^p we must firstly confine the associated twist path \mathbf{Q} to lie in $\mathcal{W}^{\lambda, p}([a, b], \mathbf{SO}(n))$ and secondly and without loss of generality set the boundary map $\varphi = \varphi(x; \mathbf{I}, \mathbf{R}) \in \mathcal{C}(\partial\mathbb{X}^n, \mathbb{S}^{n-1})$ to have the explicit form

$$\varphi(x) = \begin{cases} \mathbf{I}x|x|^{-1} & \partial\mathbb{X}_a^n = \{x : |x| = a\}, \\ \mathbf{R}x|x|^{-1} & \partial\mathbb{X}_b^n = \{x : |x| = b\}, \end{cases} \quad (10)$$

where $\mathbf{I} = \mathbf{I}_n$ is the identity matrix and $\mathbf{R} \in \mathbf{SO}(n)$ is fixed. Thus in particular any such twist path must satisfy the boundary conditions $\mathbf{Q}(a) = \mathbf{I}$, $\mathbf{Q}(b) = \mathbf{R}$. In light of this description one can calculate that for a spherical twist $u = \mathbf{Q}(r)\theta$ we have $\nabla u = (\mathbf{Q} + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta \otimes \theta)/r$ and thus (see [31] for details)

$$\mathbb{F}[\mathbf{Q}\theta] = \int_{\mathbb{X}^n} F(|\nabla[\mathbf{Q}\theta]|^2) dx = \int_{\mathbb{S}^{n-1}} \int_a^b F\left(\frac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2\right) r^{n-1} dr d\mathcal{H}^{n-1}. \quad (11)$$

In even dimensions spherical twists with a suitable choice of twist paths can be shown to yield non-trivial solutions to the nonlinear system (2). In sharp contrast, in odd dimensions, a similar analysis reveals that the only spherical twist solutions to (2) are the radial projections $u = \Pi = x|x|^{-1}$ (see [31] and below for more). As a matter of fact, commenting further, in even dimensions, the twist paths that grant spherical twist solutions u can be shown to have the form $\mathbf{Q}(r) = \exp(\mathcal{G}(r)\mathbf{H})$ where the angle of rotation function $\mathcal{G} \in \mathcal{C}^2([a, b], \mathbb{R})$ solves an associated second order ODE with $\mathcal{G}(a) = 0$, $\mathcal{G}(b) = \eta + 2\pi m$ while $-\pi \leq \eta < \pi$, $m \in \mathbb{Z}$ [see (16) below] and \mathbf{H} the $n \times n$ skew-symmetric matrix

$$\mathbf{H} = \mathbf{P} \text{diag}(\mathbf{J}, \dots, \mathbf{J}) \mathbf{P}^t, \quad \mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad (12)$$

with \mathbf{P} in $\mathbf{O}(n)$ (see, e.g., [11, 22, 31] and below). This therefore restricts the boundary map φ by requiring \mathbf{R} to have the block diagonal form [see (15)]

$$\mathbf{R} = \mathbf{P} \text{diag}(\mathcal{R}[\eta], \dots, \mathcal{R}[\eta]) \mathbf{P}^t, \quad \eta \in [-\pi, \pi). \quad (13)$$

Alternatively upon computing the matrix exponential in the above description of $\mathbf{Q}(r) = \exp(\mathcal{G}(r)\mathbf{H})$, the twist path associated with a spherical twist solution u here can be written as,

$$\mathbf{Q}(r) = \exp(\mathcal{G}(r)\mathbf{H}) = \mathbf{P} \text{diag}(\mathcal{R}[\mathcal{G}(r)], \dots, \mathcal{R}[\mathcal{G}(r)]) \mathbf{P}^t, \quad (14)$$

where $\mathcal{R}[s] = \exp(s\mathbf{J})$ is the usual 2×2 rotation matrix

$$\mathcal{R}[s] = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}. \quad (15)$$

Furthermore it follows that for the spherical twists of the form (14) one can translate (2) into an associated ODE for $\mathcal{G} = \mathcal{G}(r; m)$ given by

$$\begin{cases} \frac{d}{dr} \left[F' \left(\frac{n-1}{r^2} + \dot{\mathcal{G}}(r)^2 \right) r^{n-1} \dot{\mathcal{G}}(r) \right] = 0, & a < r < b, \\ \mathcal{G}(a) = 0, \\ \mathcal{G}(b) = \eta + 2\pi m, \end{cases} \quad (16)$$

where $-\pi \leq \eta < \pi$ and $m \in \mathbb{Z}$ [cf. (13)]. Under the stated convexity and monotonicity assumptions on F it can be shown, using variational methods, that the above two-point boundary-value problem has a unique solution $\mathcal{G} = \mathcal{G}(r; m)$ of class \mathcal{C}^2 for every $m \in \mathbb{Z}$ (cf. [11] for details).

Now in the particular case of the Dirichlet energy (with $F(t) \equiv t$) the system (2) reduces to the well-known harmonic map equation into spheres, that is,

$$\mathbf{HME}[u; \mathbb{X}^n, \mathbb{S}^{n-1}] := \begin{cases} \Delta u + |\nabla u|^2 u = 0 & \text{in } \mathbb{X}^n, \\ |u| = 1 & \text{in } \mathbb{X}^n, \\ u = \varphi & \text{on } \partial \mathbb{X}^n. \end{cases} \quad (17)$$

If a twice continuously differentiable spherical twist is a solution to (17) then it will be referred to as a *spherical harmonic twist*. Using the divergence theorem it is not difficult to see that if u is a spherical twist with a weakly differentiable twist path then the Dirichlet energy of $u = \mathbf{Q}(|x|)x|x|^{-1}$ reduces to (see [31] for details)

$$\begin{aligned} \mathbb{F}[u; \mathbb{X}^n] &= \int_{\mathbb{X}^n} |\nabla u|^2 = \int_{\mathbb{S}^{n-1}} \int_a^b \left(\frac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2 \right) r^{n-1} dr d\mathcal{H}^{n-1} \\ &= n\omega_n \int_a^b \left[\frac{n-1}{r^2} + \frac{|\dot{\mathbf{Q}}|^2}{n} \right] r^{n-1} dr, \end{aligned} \quad (18)$$

where $\omega_n = |\mathbb{B}^n|$. Thus the candidate spherical twist solutions $u = \mathbf{Q}(r)x|x|^{-1}$ to (17) will have to have their twist paths arising as extremals of this reduced energy. Analysis of the resulting ODE in conjunction with (17) then implies that for n even these spherical twists must correspond to twist paths of the form

$$\mathbf{Q}(r; m) = \exp(\mathcal{G}(r; m)\mathbf{H}) = \exp((\eta + 2\pi m)\mathcal{N}(r)\mathbf{H}), \quad m \in \mathbb{Z}, \quad (19)$$

where the angle of rotation function $\mathcal{G}(r; m) = (\eta + 2\pi m)\mathcal{N}(r)$ is the solution to (16) with the profile $\mathcal{N} = \mathcal{N}(r)$ (with $a \leq r \leq b$) being given explicitly by

$$\mathcal{N}(r) := \begin{cases} \frac{\ln r/a}{\ln b/a} & \text{for } n = 2, \\ \frac{(r/a)^{2-n} - 1}{(b/a)^{2-n} - 1} & \text{for } n > 2. \end{cases} \quad (20)$$

A careful analysis using (17) and (19) then reveals that for n even every such $\mathbf{Q} = \mathbf{Q}(r; m)$ results in a spherical harmonic twist $u(x) = \mathbf{Q}(|x|; m)x|x|^{-1}$. In sharp contrast for n odd this can happen only when $\mathbf{Q} \equiv \mathbf{I}$ and therefore the only spherical harmonic twists here are the radial projections $u(x) = \Pi(x) = x|x|^{-1}$. (See [31] for

details.) Now to discuss the stability and local minimising properties of these spherical harmonic twists we move on to the second variation of the energy and examining its positivity.²

The second variation of the Dirichlet energy at the spherical harmonic twist $u(x) = \mathbf{Q}(|x|; m)x|x|^{-1}$, with \mathbf{Q} as in (19), in the direction $\phi \in \mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$ and with $\hat{\phi} = D\Pi(u)\phi = \phi - \langle \phi, u \rangle u$ denoting the tangential part of the variation ϕ , is seen to be, (see Appendix for derivation)

$$\left. \frac{d^2}{d\varepsilon^2} \mathbb{F}[\Pi(u + \varepsilon\phi)] \right|_{\varepsilon=0} = 2 \int_{\mathbb{X}^n} \left(|\nabla \hat{\phi}|^2 - \left[\frac{n-1}{r^2} + \mathcal{N}^2(\eta + 2\pi m)^2 \right] |\hat{\phi}|^2 \right) dx. \quad (21)$$

We find that this second variation is positive definite providing $\zeta = \eta + 2\pi m$ is smaller than a bound hence putting restraints on the twist path $r \mapsto \mathbf{Q}(r)$. The analysis relates to examining the restricted energy and studying the Jacobi fields and conjugate points along the twist paths $\mathbf{Q} = \mathbf{Q}(r)$ on $\mathbf{SO}(n)$. We obtain explicitly the conjugate points by invoking the root system of the compact Lie group $\mathbf{SO}(n)$ and its Lie algebra $\mathfrak{so}(n)$. With these at hand we then move on to proving the main result of the paper, giving conditions on a *spherical harmonic twist* to be a strong local energy minimiser. To achieve this we prove that under a suitable bound on the twist path there exists $\gamma > 0$ such that

$$\text{RHS (21)} \geq \gamma \|\hat{\phi}\|_{\mathcal{W}^{1,2}}^2, \quad \forall \hat{\phi} \in \mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n) : \langle \hat{\phi}, u \rangle = 0, \quad (22)$$

and then translate this, by considering and analysing a modified energy involving the radial projection Π , into the desired minimality result.

Main Theorem (Dirichlet energy) *Suppose $n \geq 3$ and consider the spherical harmonic twist $u(x) = \mathbf{Q}(|x|)x|x|^{-1}$ with $x \in \mathbb{X}^n$ as solution to the nonlinear system HME (17). Then depending on n being odd or even we have*

- (n odd) $\mathbf{Q} \equiv \mathbf{I}_n$ and $\mathbb{F}[u \equiv x|x|^{-1}; \mathbb{X}^n] \leq \mathbb{F}[v; \mathbb{X}^n] \forall v \in \mathcal{W}^{1,2}(\mathbb{X}^n, \mathbb{S}^{n-1})$ with $v = u$ on $\partial\mathbb{X}^n$.
- (n even) $\mathbf{Q}(r) = \exp(\mathcal{G}(r; m)\mathbf{H}) = \exp(\zeta \mathcal{M}(r)\mathbf{H})$ with $a \leq r \leq b$, $m \in \mathbb{Z}$, $\zeta = \eta + 2\pi m$, and \mathbf{H} as in (12). Furthermore we have the implication

$$|\zeta| = \|\mathcal{G} = \zeta \mathcal{M}\|_{L^\infty(a,b)} \leq \frac{n-4}{2n-4} (1 - (a/b)^{n-2}) \implies (22).$$

² For $n \geq 3$ the radial projection $u = x|x|^{-1}$ is known to be the minimiser of the Dirichlet energy (i.e., $F(t) \equiv t$) over $\mathcal{A}_\varphi^\mathcal{L}(\mathbb{B})$ with $\varphi = x|x|^{-1}$ on $\partial\mathbb{B}$ (see, e.g., [20] and for the related and more involved setting of weighted and p -energies see [6] and the list of references therein). A basic argument involving "filling the hole" now leads to a similar minimality conclusion over $\mathcal{A}_\varphi^\mathcal{L}(\mathbb{X})$. In view of this remark and the characterisation of spherical harmonic twists for n odd as radial projections there will be no loss of generality in restricting hereafter the investigation of stability and local minimality to $n \geq 2$ even. (See also the comments preceding Theorem 2.)

Therefore here $\exists \varepsilon > 0$ such that $\forall v \in \mathcal{W}^{1,2}(\mathbb{X}^n, \mathbb{S}^{n-1})$ with $v = u$ on $\partial \mathbb{X}^n$, $0 < \|u - v\|_{L^1} < \varepsilon \implies \mathbb{F}[u; \mathbb{X}^n] < \mathbb{F}[v; \mathbb{X}^n]$.

Let us finish off this introduction by making some comments on the theorem. Firstly for n odd, by part one of the theorem, the only spherical twist solution to **HME** (17) is the radial projection $u = x|x|^{-1}$ in which case necessarily we must have $\mathbf{R} = \mathbf{I}$ in (10). For n even, by contrast, there is a countably infinite family of twisting solutions to (17) all having the form $u = \mathbf{Q}(r; m)x|x|^{-1}$ ($m \in \mathbb{Z}$) where \mathbf{Q} is as described in the second part of the theorem. Next the bound on $|\zeta|$ as given in the second part of the theorem implies the positivity of the second variation of the energy, that is, (22) regardless of $n \geq 4$ being odd or even. However its application to strong local minimality of spherical harmonic twists is mainly of interest for n even as for n odd the radial projection is a global energy minimiser. Thirdly the main strength of the theorem in the n even case comes from $n \geq 6$ as for $n = 4$ the stated bound applies to the radial projection (with $\zeta = 0$) only. Fourthly by studying Jacobi fields and conjugate points on $\mathbf{SO}(n)$ ($n \geq 3$) in Sect. 2 it will be shown that the second variation of the Dirichlet energy restricted to the space of twist paths [cf. 23] is positive definite at the stationary twist path $\mathbf{Q}(r) = \zeta \mathcal{M}(r)\mathbf{H}$ if only the weaker condition $|\zeta| < \pi$ holds (compare with the bound on $|\zeta|$ in part two of the theorem). This has the interesting implication that for $|\zeta| > \pi$ the spherical harmonic twist u is neither globally nor locally energy minimising as there would be variations in the form of spherical twists that minimise energy. Now since the Dirichlet energy certainly attains its infimum over the space $\mathcal{S}_\varphi^2(\mathbb{X})$ for φ as in (10) it remains to be seen if the spherical harmonic twist $u = \exp(\zeta \mathcal{M}(|x|)\mathbf{H})x|x|^{-1}$ is globally minimising when $|\zeta|$ satisfies the bound in the second part of the theorem. Finally for $n = 2$ and other related results we refer the reader to the appendix at the end.

2 The second variation of the reduced energy on $\mathbf{SO}(n)$: Conjugate points and Jacobi fields

In this section we examine the \mathbb{G} -energy of twist paths $\mathbf{Q} = \mathbf{Q}(r)$ resulting from restricting the Dirichlet energy [that is, (1) with $F(t) \equiv t$] to the space of spherical twists. This is given explicitly by the integral [compare with (18)]

$$\mathbb{G}[\mathbf{Q}; \mathbf{SO}(n)] := \int_a^b \text{tr}\{[\dot{\mathbf{Q}}]^t[\dot{\mathbf{Q}}]\} r^{n-1} dr, \quad \dot{\mathbf{Q}} = d\mathbf{Q}/dr. \quad (23)$$

The aim is to seek conditions on a stationary path $\mathbf{Q} = \mathbf{Q}(r)$ to ensure that the second variation of the \mathbb{G} -energy at \mathbf{Q} is positive. To simplify presentation from here on we assume $n \geq 3$, however, similar analysis can be carried out in the case $n = 2$ [cf. (20)]. Now put $\mathcal{B} := \{\mathbf{Q} \in \mathcal{W}^{1,2}([a, b], \mathbf{SO}(n)) : \mathbf{Q}(a) = \mathbf{I}_n\}$ and fix a stationary path $\mathbf{Q} \in \mathcal{B}$ of the \mathbb{G} -energy. (Recall that boundary values of Sobolev maps are interpreted in the usual sense of traces.) By taking proper variations Γ_ε of \mathbf{Q} such

that $d\Gamma/d\varepsilon|_{\varepsilon=0} = \mathbf{M}\mathbf{Q}$ for some $\mathbf{M} \in \mathcal{C}_0^\infty([a, b], \mathbb{M}_{skew}^{n \times n})$ one can express the second variation of the \mathbb{G} -energy at \mathbf{Q} in the tangential direction $\mathbf{M}\mathbf{Q}$ by

$$\begin{aligned} \delta^2 \mathbb{G}[\mathbf{Q}; \mathbf{SO}(n)](\mathbf{M}, \mathbf{M}) &= \frac{d^2}{d\varepsilon^2} \mathbb{G}[\Gamma_\varepsilon; \mathbf{SO}(n)]|_{\varepsilon=0} \\ &= 2 \int_a^b [|D_r(\mathbf{M}\mathbf{Q})|^2 + \langle \dot{\mathbf{Q}}, R(\mathbf{M}\mathbf{Q}, \dot{\mathbf{Q}})\mathbf{M}\mathbf{Q} \rangle] r^{n-1} dr. \end{aligned} \quad (24)$$

Here R is the Riemann curvature tensor and D_r denotes the covariant derivative with respect to r (see [10, 26]).³ It is not hard to see that the stationary paths of the \mathbb{G} -energy are given by the matrix exponential $\mathbf{Q}(r) = \exp(\mathcal{N}(r)\mathbf{A})$ with \mathbf{A} skew-symmetric and \mathcal{N} as in (20) (see [31]). We now seek conditions on these stationary paths $\mathbf{Q} = \mathbf{Q}(r)$ to ensure that the second variation of the \mathbb{G} -energy (23) at \mathbf{Q} is positive definite. In order to do so we study a suitably adapted form of Jacobi fields and conjugate points for the \mathbb{G} -energy on the compact Lie group $\mathbf{SO}(n)$. Indeed here by a Jacobi field along \mathbf{Q} we mean a continuously differentiable vector field $\xi = \xi(r)$ defined on the stationary path $\mathbf{Q} = \mathbf{Q}(r)$ that is a solution to the second order linear ODE⁴

$$D_r(r^{n-1}D_r\xi) + R(\xi, \dot{\mathbf{Q}})\dot{\mathbf{Q}}r^{n-1} = 0, \quad a < r < b. \quad (25)$$

Now $\mathbf{F} \in \mathbf{SO}(n)$ is said to be conjugate to \mathbf{I}_n along $\mathbf{Q} = \mathbf{Q}(r)$ iff $\mathbf{F} = \mathbf{Q}(\rho)$ for some $\rho > a$ and there exists a Jacobi field $\xi \not\equiv 0$ along \mathbf{Q} that vanishes at a and ρ . The study of the conjugate points here is deeply intertwined with the integral (index form)

$$\mathcal{J}(\mathbf{M}, \mathbf{N}) = \int_a^b \{ \langle D_r(\mathbf{M}\mathbf{Q}), D_r(\mathbf{N}\mathbf{Q}) \rangle + \langle \dot{\mathbf{Q}}, R(\mathbf{M}\mathbf{Q}, \dot{\mathbf{Q}})\mathbf{N}\mathbf{Q} \rangle \} r^{n-1} dr.$$

Note that $2\mathcal{J}(\mathbf{M}, \mathbf{M})$ is exactly the second variation of the \mathbb{G} -energy at \mathbf{Q} . Now the connection between conjugate points and the positivity of the second variation is classical and lies in the fact that if $\mathbf{Q} \in \mathcal{B}$ is a stationary path of the \mathbb{G} -energy and if $\mathbf{Q}(a)$ has no conjugate points along \mathbf{Q} on $(a, b]$ then \mathcal{J} is positive definite on the space of piecewise \mathcal{C}^1 vector fields \mathbf{Y} along \mathbf{Q} vanishing at $r = a$ and b . (For more background on this and further references *cf.*, e.g., [3] or [10] Theorem II.5.4). A stationary path $\mathbf{Q} = \mathbf{Q}(r)$ with $a \leq r \leq b$ is locally minimising (among nearby paths) if there are no conjugate points to $r = a$ along \mathbf{Q} at $\rho \in]a, b]$ but it *fails* to be locally minimising past its first conjugate point. We therefore set ourselves the

³ The reader is particularly reminded about the different sign conventions for the curvature tensor as used by authors and in literature.

⁴ Here we have modified the usual definition of a Jacobi field to account for the weight r^{n-1} in (23). Indeed a suitable change of scale transforms the \mathbb{G} -energy (23) into the Dirichlet energy of \mathbf{Q} and subsequently an application of the same change of scale to the Jacobi equation along geodesics gives (25). For more on Jacobi fields, conjugate points and their relation to the positivity of the second variation and local energy minimality see [3, 9, 10] and the references therein.

task of computing conjugate points to the identity $\mathbf{Q}(a) = \mathbf{I}_n$ along certain stationary paths of the \mathbb{G} -energy in \mathcal{B} . Here the path $r \mapsto \mathbf{Q}(r) = \exp(\mathcal{N}(r)\mathbf{A})$ will be taken on the infinite interval $[a, \infty)$ and to compute conjugate points we use some elements of the representation theory of compact Lie groups (for a nice introduction see [17, 24, 25]). Note in particular that the paths $\mathbf{Q} = \mathbf{Q}(r)$ considered here specifically include those twist paths associated with spherical harmonic twists $u = \mathbf{Q}(|x|)x|x|^{-1}$ discussed earlier in Sect. 1. We exploit this connection further in Sect. 3.

Proposition 1 *Assume $n \geq 3$ and depending on n being even or odd suppose $\mathbf{A} \in \mathfrak{so}(n)$ is the block diagonal skew-symmetric matrix*

$$\mathbf{A}(\zeta_1, \dots, \zeta_k) = \begin{cases} \text{diag}(\zeta_1 \mathbf{J}, \dots, \zeta_k \mathbf{J}) & \text{if } n = 2k, \\ \text{diag}(\zeta_1 \mathbf{J}, \dots, \zeta_k \mathbf{J}, 0) & \text{if } n = 2k + 1, \end{cases} \quad (26)$$

with $(\zeta_\sigma : 1 \leq \sigma \leq k) \subset \mathbb{R}$. Put $\mathbf{Q}(r) = \exp(\mathcal{N}(r)\mathbf{A})$ with $a \leq r < \infty$ and \mathcal{N} as in (20). Then the conjugate points to $\mathbf{Q}(a) = \mathbf{I}_n$ along $\mathbf{Q} = \mathbf{Q}(r)$ are given by the collection of $\mathbf{SO}(n)$ matrices \mathbf{F} as formulated below:

- ($n = 2k$) Here $\mathbf{F}_{\sigma\tau}^\pm(m) = \mathbf{Q}(\rho_{\sigma\tau}^\pm(m))$ for $1 \leq \sigma < \tau \leq k$,
- ($n = 2k + 1$) Here $\mathbf{F}_{\sigma\tau}^\pm(m) = \mathbf{Q}(\rho_{\sigma\tau}^\pm(m))$ for $1 \leq \sigma < \tau \leq k$ and $\hat{\mathbf{F}}_\sigma(m) = \mathbf{Q}(\hat{\rho}_\sigma(m))$ for $1 \leq \sigma \leq k$.

Note that in either of the cases $0 \leq m = m_{\sigma,\tau} < |\zeta_\sigma \pm \zeta_\tau| / \{2\pi[1 - (b/a)^{2-n}]\}$ and $0 \leq m = m_\sigma < |\zeta_\sigma| / \{2\pi[1 - (b/a)^{2-n}]\}$ while the quantities $\rho_{\sigma\tau}^\pm(m)$ and $\hat{\rho}_\sigma(m)$ are given respectively by

$$\rho_{\sigma\tau}^\pm(m) = \mathcal{N}^{-1}\left(\frac{2\pi m}{|\zeta_\sigma \pm \zeta_\tau|}\right) = a \left\{ 1 + 2\pi m \frac{(b/a)^{2-n} - 1}{|\zeta_\sigma \pm \zeta_\tau|} \right\}^{\frac{1}{2-n}}, \quad (27)$$

$$\hat{\rho}_\sigma(m) = \mathcal{N}^{-1}\left(\frac{2\pi m}{|\zeta_\sigma|}\right) = a \left\{ 1 + 2\pi m \frac{(b/a)^{2-n} - 1}{|\zeta_\sigma|} \right\}^{\frac{1}{2-n}}. \quad (28)$$

Additionally if $\zeta_\sigma = 0$ or $|\zeta_\sigma \pm \zeta_\tau| = 0$, then the conjugate points are at infinity and so we omit the associated points $\mathbf{F}_{\sigma\tau}^\pm(m)$ and $\hat{\mathbf{F}}_\sigma(m)$ from the list.

Proof On a Lie group \mathbf{G} the tangent bundle is trivial (\mathbf{G} is parallelisable) and the Riemann curvature endomorphism is given, for left invariant vector fields, $X, Y, Z \in \mathfrak{g}$ by the Lie bracket

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]]. \quad (29)$$

In case of $\mathbf{G} = \mathbf{SO}(n)$ we have $[X, Y] = XY - YX$ and the bi-invariant metric on \mathbf{G} results from Killing form $B(X, Y) = (n - 2)\text{tr}(XY)$ with $X, Y \in \mathfrak{g} = \mathfrak{so}(n)$.

(For the related jargon on curvature, geometry and representation of Lie groups as used here the reader is referred to [17, 19, 24].) Now given \mathbf{A} as in (26) the path $\mathbf{Q}(r) = \exp(\mathcal{N}(r)\mathbf{A})$ takes values on the maximal torus $\mathbb{T} \subset \mathbf{SO}(n)$ of block diagonal rotation matrices given by

$$\mathbb{T} := \begin{cases} \{ \text{diag}(\mathcal{R}[\theta_1], \dots, \mathcal{R}[\theta_k]) : \theta_1, \dots, \theta_k \in \mathbb{R} \} & \text{if } n = 2k, \\ \{ \text{diag}(\mathcal{R}[\theta_1], \dots, \mathcal{R}[\theta_k], 1) : \theta_1, \dots, \theta_k \in \mathbb{R} \} & \text{if } n = 2k + 1. \end{cases}$$

Let $\mathfrak{t} \subset \mathfrak{so}(n)$ denote the Lie algebra tangent to the maximal torus $\mathbb{T} \subset \mathbf{SO}(n)$, $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t}$ the Cartan subalgebra associated with \mathbb{T} and $\Delta = \Delta(\mathfrak{so}(n)) \subset \mathfrak{h}^*$ the finite set of roots with \mathfrak{h}^* being the dual space to \mathfrak{h} . Starting from the root space decomposition $\mathfrak{so}(n)_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus_{\alpha \in \Delta} [\mathfrak{so}(n)_{\mathbb{C}}]_{\alpha}$ with $[\mathfrak{so}(n)_{\mathbb{C}}]_{\alpha}$ denoting the one dimensional root space associated with the root α , specifically, $[\mathfrak{so}(n)_{\mathbb{C}}]_{\alpha} = \{ \mathbf{Y} \in \mathfrak{so}(n)_{\mathbb{C}} : \text{ad}_{\mathbf{A}}(\mathbf{Y}) = [\mathbf{A}, \mathbf{Y}] = \alpha(\mathbf{A})\mathbf{Y} \text{ for all } \mathbf{A} \in \mathfrak{h} \}$ it is seen that

$$-\text{ad}_{\mathbf{A}}^2[\mathbf{Y}_{\alpha}] = [\mathbf{A}, [\mathbf{Y}_{\alpha}, \mathbf{A}]] = [\mathbf{A}, -\alpha(\mathbf{A})\mathbf{Y}_{\alpha}] = |\alpha(\mathbf{A})|^2 \mathbf{Y}_{\alpha}, \quad (30)$$

for $\mathbf{Y}_{\alpha} \in [\mathfrak{so}(n)_{\mathbb{C}}]_{\alpha}$, $\mathbf{A} \in \mathfrak{t}$, by noting that $\alpha(\mathbf{A})$ is purely imaginary for $\mathbf{A} \in \mathfrak{t}$. Thus by linearity in \mathbf{Y} the latter identity continues to hold for any \mathbf{Y} in the linear span of $[\mathfrak{so}(n)_{\mathbb{C}}]_{\pm\alpha}$. (For an explicit description of these spaces in the case of $\mathbf{SO}(n)$ see [17] pp. 40–1, 78–80.) This leads to the real root space decomposition $\mathfrak{so}(n) = \mathfrak{t} \oplus_{\alpha \in \Delta^+} \mathfrak{so}(n)_{\alpha}$ where Δ^+ is the set of positive roots and each $\mathfrak{so}(n)_{\alpha}$ is a two dimensional real subspace [eigen-space] of $[\mathfrak{so}(n)_{\mathbb{C}}]_{\alpha}$ on which (30) holds. Hence for $\mathbf{Y}_{\alpha} \in \mathfrak{so}(n)_{\alpha}$, by recalling (29) we have $4R(\mathbf{Y}_{\alpha}, \mathbf{A})\mathbf{A} = |\alpha(\mathbf{A})|^2 \mathbf{Y}_{\alpha}$. Let $\mathbf{Q}(r) = \exp(\mathcal{N}(r)\mathbf{A})$ and let $E_{\alpha}(r) = \mathbf{Y}_{\alpha}\mathbf{Q}$ be the parallel vector field along \mathbf{Q} such that $E_{\alpha}(a) = \mathbf{Y}_{\alpha}$, with $D_r E_{\alpha} = 0$. Then, setting $\xi_{\alpha}(r) = u_{\alpha}(r)E_{\alpha}(r)$, and referring to (25) we can write

$$\begin{aligned} J &= D_r(r^{n-1}D_r\xi_{\alpha}) + R(\xi_{\alpha}, \dot{\mathbf{Q}})\dot{\mathbf{Q}}r^{n-1} \\ &= \frac{d}{dr}(\dot{u}_{\alpha}r^{n-1})E_{\alpha} + u_{\alpha}\dot{\mathcal{N}}^2 R(E_{\alpha}, \mathbf{A}\mathbf{Q})\mathbf{A}\mathbf{Q}r^{n-1} \\ &= \frac{d}{dr}(\dot{u}_{\alpha}r^{n-1})E_{\alpha} + u_{\alpha}\dot{\mathcal{N}}^2 (R(\mathbf{Y}_{\alpha}, \mathbf{A})\mathbf{A})\mathbf{Q}r^{n-1} \\ &= \frac{d}{dr}(\dot{u}_{\alpha}r^{n-1})E_{\alpha} + u_{\alpha}\dot{\mathcal{N}}^2 \frac{|\alpha(\mathbf{A})|^2}{4} E_{\alpha}r^{n-1}. \end{aligned} \quad (31)$$

It therefore follows that the vector field $\xi_{\alpha}(r) = u_{\alpha}(r)E_{\alpha}(r)$ is a Jacobi field along \mathbf{Q} (i.e., $J = 0$) iff u_{α} satisfies the equation

$$\ddot{u}_{\alpha} + \frac{n-1}{r}\dot{u}_{\alpha} + \frac{|\alpha(\mathbf{A})|^2}{4}\mathcal{N}(r)^2 u_{\alpha} = 0. \quad (32)$$

Using the change of variables $s \mapsto (r^{2-n}|\alpha(\mathbf{A})|)/(2(b^{2-n} - a^{2-n}))$ this becomes the ODE $\partial_{ss}u_{\alpha} + u_{\alpha} = 0$. Hence solutions to this satisfying $u_{\alpha}(a) = 0$ can be computed

explicitly. Now for each $\mathbf{Y} \in \mathfrak{so}(n)$ there is a unique Jacobi field satisfying the initial conditions $\xi(a) = 0$ and $D_r \xi(a) = \mathbf{Y}$. Since for $\mathbf{Y} \in \mathfrak{t}$ we have $R(\mathbf{Y}, \mathbf{A})\mathbf{A} = 0$ and so the resulting Jacobi field is either trivial or vanishes in one point, in describing conjugate points it is enough to focus on $\mathbf{Y} \in \mathfrak{so}(n)_\alpha$. Indeed any such field along \mathbf{Q} vanishing at $r = a$ is of the form

$$\begin{aligned} \xi(r) &= \sum_{\alpha \in \Delta^+} u_\alpha(r) E_\alpha(r) = \sum_{\alpha \in \Delta^+} c_\alpha \sin \left(\frac{|\alpha(\mathbf{A})|}{2} \frac{r^{2-n} - a^{2-n}}{b^{2-n} - a^{2-n}} \right) E_\alpha(r) \\ &= \sum_{\alpha \in \Delta^+} c_\alpha \sin \left(\frac{|\alpha(\mathbf{A})|}{2} \mathcal{N}(r) \right) E_\alpha(r) \end{aligned} \quad (33)$$

for constants $c_\alpha \in \mathbb{R}$. We can therefore see that the zeros here occur only when $|\alpha(\mathbf{A})| \mathcal{N}(r) = 2m\pi$ with m integer. Now considering the range of \mathcal{N} and its inverse \mathcal{N}^{-1} and noting $a \leq r < \infty$ this restricts the values of m to the interval $0 \leq m < |\alpha(\mathbf{A})|/2\pi[1 - (b/a)^{2-n}]$ and gives

$$r = \mathcal{N}^{-1} \left(\frac{2m\pi}{|\alpha(\mathbf{A})|} \right) = \left\{ a^{2-n} + \frac{2m\pi}{|\alpha(\mathbf{A})|} (b^{2-n} - a^{2-n}) \right\}^{\frac{1}{2-n}}. \quad (34)$$

Thus to complete the description of the conjugate points it suffices to substitute for the roots $\alpha \in \Delta$ in (34). By standard results (see, e.g., [17] pp. 78–80 or [24] pp. 122–3) and with \mathbf{A} as given by (26) these are:

- ($n = 2k$) Here $\alpha(\mathbf{A}) = \pm\sqrt{-1}(\zeta_\sigma \pm \zeta_\tau)$ with $1 \leq \sigma < \tau \leq k$.
- ($n = 2k + 1$) Here $\alpha(\mathbf{A}) = \pm\sqrt{-1}(\zeta_\sigma \pm \zeta_\tau)$ with $1 \leq \sigma < \tau \leq k$ or $\alpha(\mathbf{A}) = \pm\sqrt{-1}\zeta_\sigma$ with $1 \leq \sigma \leq k$.

We can now explicitly describe all conjugate points to the identity along \mathbf{Q} via (34). This therefore completes the proof. \square

Theorem 1 *Let $n \geq 3$ and consider the stationary path $\mathbf{Q}(r) = \exp(\mathcal{N}(r)\mathbf{A})$ where $r \geq a$, \mathcal{N} is as in (20) and \mathbf{A} is as in (26). Assume additionally that*

- ($n = 2k$) $1 < \min\{2\pi/|\zeta_\sigma \pm \zeta_\tau| : 1 \leq \sigma < \tau \leq k\}$.
- ($n = 2k + 1$) $1 < \min\{2\pi/|\zeta_\sigma \pm \zeta_\tau| : 1 \leq \sigma < \tau \leq k, \quad 2\pi/|\zeta_\sigma| : 1 \leq \sigma \leq k\}$.

Then the second variation of the \mathbb{G} -energy (23) is positive at \mathbf{Q} , i.e., the strict inequality $\delta^2 \mathbb{G}[\mathbf{Q}; \mathbf{SO}(n)](\mathbf{M}, \mathbf{M}) > 0$ holds for all non-zero skew-symmetric matrix fields $\mathbf{M} \in \mathcal{C}_0^\infty([a, b], \mathbb{M}_{skew}^{n \times n})$.

Proof The second variation of \mathbb{G} is positive at \mathbf{Q} if there are no conjugate points to $\mathbf{Q}(a) = \mathbf{I}_n$ along \mathbf{Q} in $(a, b]$. Now the calculations in Proposition 1 reveal that the first conjugate point to \mathbf{I}_n along \mathbf{Q} occurs at $\mathbf{Q}(\rho)$ where

$$\rho = \begin{cases} \min \{ \rho_{\sigma\tau}^{\pm}(1) : 1 \leq \sigma < \tau \leq k \} & \text{if } n = 2k, \\ \min \{ \rho_{\sigma\tau}^{\pm}(1) : 1 \leq \sigma < \tau \leq k, \quad \hat{\rho}_{\sigma}(1) : 1 \leq \sigma \leq k \} & \text{if } n = 2k + 1. \end{cases}$$

Therefore if $\rho > b$ then there will be no conjugate points to \mathbf{I}_n along \mathbf{Q} on $(a, b]$. The conclusion of the theorem now follows by substituting the expressions for $\rho_{\sigma\tau}^{\pm}(1)$, $\hat{\rho}_{\sigma}(1)$ and rearranging. \square

Remark 1 Any closed geodesic $\mathbf{Q} = \mathbf{Q}(t)$ ($0 \leq t \leq 1$) on $\mathbf{SO}(n)$ (with $n \geq 3$) based at identity \mathbf{I}_n has the form $\mathbf{Q}(t) = \mathbf{P} \exp(t\mathbf{A}) \mathbf{P}^t$ where \mathbf{A} is as in (26), $\zeta_j = 2\pi m_j$, $m_j \in \mathbb{Z}$ for $1 \leq j \leq k$ and $\mathbf{P} \in \mathbf{O}(n)$. Here $\pi_1[\mathbf{SO}(n)] \cong \mathbb{Z}_2$ and depending on the parity of $m = m_1 + \dots + m_k$ being even or odd \mathbf{Q} represents the trivial or non-trivial element of $\pi_1[\mathbf{SO}(n)]$. When all but one of the m_j s are zero and $m = \pm 1$ the resulting geodesic is energy minimising in its homotopy class and hence locally energy minimising. It is seen that here the only non-zero root value verifies $|\alpha(\mathbf{A})| = 2\pi$ and so by the argument in Proposition 1 the first conjugate point to \mathbf{I}_n along \mathbf{Q} occurs at $t = 1$ (a complete closed loop) in line with the minimality of \mathbf{Q} . The only other energy minimising geodesic occurs now when all m_j are zero $\mathbf{Q}(t) \equiv \mathbf{I}_n$ and this is globally minimising.

Proceeding forward now and with a view towards comparing the positivity of the second variations of two closely related energies, namely, the \mathbb{G} -energy (23) at a stationary twist path $\mathbf{Q} = \mathbf{Q}(r)$ and the \mathbb{F} -energy (18) at the associated spherical harmonic twist $u = \mathbf{Q}(|x|)x|x|^{-1}$, we next introduce a naturally arising third energy for weakly differentiable curves on \mathbb{S}^{n-1} by the weighted integral

$$\mathbb{G}[\gamma; \mathbb{S}^{n-1}] := \int_a^b |\dot{\gamma}|^2 r^{n-1} dr, \quad \dot{\gamma} = d\gamma/dr, \gamma \in \mathcal{W}^{1,2}([a, b], \mathbb{S}^{n-1}). \quad (35)$$

Aiming to understand the form and structure of energy minimisers, it is seen that the stationary curves γ satisfy the associated Euler–Lagrange equation, here the second order ODE,

$$\ddot{\gamma} + \frac{n-1}{r} \dot{\gamma} + |\dot{\gamma}|^2 \gamma = 0, \quad a < r < b. \quad (36)$$

Moreover by going a step further the resulting second variation of this \mathbb{G} -energy at any such stationary curve γ is described by the quadratic form

$$\delta^2 \mathbb{G}[\gamma; \mathbb{S}^{n-1}](\hat{\phi}, \hat{\phi}) = \frac{d^2}{d\varepsilon^2} \mathbb{G}[\pi(\gamma + \varepsilon\phi)] \Big|_{\varepsilon=0} = 2 \int_a^b \left[|\hat{\phi}|^2 - |\dot{\gamma}|^2 |\hat{\phi}|^2 \right] r^{n-1} dr, \quad (37)$$

where $\phi \in \mathcal{C}_0^\infty([a, b], \mathbb{R}^n)$, $\hat{\phi} = \phi - \langle \gamma, \phi \rangle \gamma$ and $\hat{\dot{\phi}} = d\hat{\phi}/dr$. Now upon integrating the ODE (36) we have the following characterisation of its solutions.

Lemma 1 Let $\gamma \in \mathcal{W}^{1,2}([a, b[, \mathbb{S}^{n-1})$ be a stationary curve of the \mathbb{G} -energy (35), that is, it is a twice continuously differentiable solution to (36). Then $|\dot{\gamma}|^2 r^{2n-2} \equiv c^2$ for some constant c and all $a < r < b$.

Proof Let γ as described satisfy (36). Then by a direct differentiation we can write $d|\dot{\gamma}|^2/dr = 2\langle \dot{\gamma}, \ddot{\gamma} \rangle = -2/r\langle \dot{\gamma}, (n-1)\dot{\gamma} \rangle - 2|\dot{\gamma}|^2\langle \dot{\gamma}, \gamma \rangle = -2(n-1)/r|\dot{\gamma}|^2$ where in deducing the last equality we have used the fact that γ takes values on the unit sphere, i.e., $|\gamma|^2 = 1$. The assertion now follows by integration. \square

Using the fact that the round sphere has constant scalar curvature combined with an analysis similar to that given earlier in the section we prove the following statement.

Proposition 2 Let $n \geq 3$ and suppose γ is a stationary curve of the \mathbb{G} -energy (35) with $|\dot{\gamma}|^2 = c^2 r^{2(1-n)}$ by Lemma 1. Assume $|c| < \pi(n-2)/(a^{2-n} - b^{2-n})$. Then the \mathbb{G} -energy has a positive second variation at γ , that is, for all non-zero $\phi \in \mathcal{C}_0^\infty([a, b[, \mathbb{R}^n)$ we have $\delta^2 \mathbb{G}[\gamma](\hat{\phi}, \hat{\phi}) > 0$ where $\hat{\phi} = \phi - \langle \gamma, \phi \rangle \gamma$.

Proof The \mathbb{G} -energy (35) has a positive second variation at γ if there are no conjugate points to $\gamma(a)$ along γ in $(a, b]$, that is, there are no non-trivial vector fields ξ along γ satisfying for $a < \beta \leq b$ the equation

$$\begin{cases} D_r(r^{n-1}D_r\xi) + R(\xi, \dot{\gamma})\dot{\gamma}r^{n-1} = 0, \\ \xi(a) = 0, \\ \xi(\beta) = 0. \end{cases} \quad (38)$$

Here R denotes the Riemann curvature tensor of \mathbb{S}^{n-1} where by virtue of \mathbb{S}^{n-1} having constant sectional curvature one, we have $R(\xi, \dot{\gamma})\dot{\gamma} = \langle \dot{\gamma}, \dot{\gamma} \rangle \xi - \langle \dot{\gamma}, \xi \rangle \dot{\gamma}$. Thus arguing in the usual way, by taking a normal vector field $\xi(r) = u(r)E(r)$ along γ with $E(r)$ parallel and the coefficient $u = u(r)$ to be specified, it is seen that

$$D_r(r^{n-1}D_r\xi) + R(\xi, \dot{\gamma})\dot{\gamma}r^{n-1} = \left(\ddot{u} + \frac{n-1}{r}\dot{u} + |c|^2 r^{2(1-n)}u \right) E r^{n-1}. \quad (39)$$

Referring to (38) it is thus seen that a normal Jacobi field along γ vanishing at $r = a$ is a linear combination of normal Jacobi fields given explicitly by $\xi(r) = A \sin(|c|(a^{2-n} - r^{2-n})/(n-2))E(r)$ with $A \in \mathbb{R}$.⁵

As conjugate points can now be computed by finding the zeros of the Jacobi field ξ , i.e., solving $\xi(r) = 0$, it follows that subject to the given bound on $|c|$ in the proposition there are no conjugate points along γ to a and so the conclusion follows. \square

⁵ Here we can restrict to normal Jacobi fields only as firstly any Jacobi field can be written as a linear combination of a normal Jacobi field and the tangential ones $\dot{\gamma}, r^{2-n}\dot{\gamma}$ and secondly the Jacobi field $(A_1 + A_2 r^{2-n})\dot{\gamma}$ is either trivial (i.e., $\equiv 0$) or has at most one zero (at $r = a$).

It is a straightforward calculation to verify that for n even and with $\mathbf{A} = \text{diag}(\zeta \mathbf{J}, \dots, \zeta \mathbf{J})$ the smooth curve $r \mapsto \gamma(r) = \exp(\mathcal{M}(r)\mathbf{A})\theta$ with $a \leq r \leq b$ is a stationary curve of (35) for any fixed $\theta \in \mathbb{S}^{n-1}$ and $\zeta \in \mathbb{R}$. Using Theorem 1 we see that on the one hand the associated twist path $r \mapsto \mathbf{Q}(r) = \exp(\mathcal{M}(r)\mathbf{A})$ has a positive second variation if $|\zeta| < \pi$. On the other hand upon considering the curve $\gamma(r) = \exp(\mathcal{M}(r)\mathbf{A})\theta$ it is seen that

$$|\dot{\gamma}(r)|^2 = \dot{\mathcal{M}}(r)^2 |\mathbf{A}\mathbf{Q}(r)\theta|^2 = \dot{\mathcal{M}}(r)^2 |\zeta|^2 = \frac{(n-2)^2 |\zeta|^2}{(a^{2-n} - b^{2-n})^2} r^{2-2n} = |c|^2 r^{2-2n},$$

and so by invoking Proposition 2 it follows that the energy has a positive second variation at γ if

$$a^{2-n} - b^{2-n} < \frac{(n-2)}{|c|} \pi = (n-2) \pi \frac{a^{2-n} - b^{2-n}}{|\zeta|(n-2)} \quad (40)$$

which upon rearranging yields $|\zeta| < \pi$. Collecting the above calculations we have the following nice proposition.

Proposition 3 *Let $\mathbf{Q}(r) = \exp(\mathcal{M}(r)\mathbf{A})$ where $\mathbf{A} = \text{diag}(\zeta \mathbf{J}, \dots, \zeta \mathbf{J})$ and $|\zeta| < \pi$. The \mathbb{G} -energies (23) and (35) have positive definite second variations at $\mathbf{Q} = \mathbf{Q}(r)$ and $\gamma = \gamma(r) = \mathbf{Q}(r)\theta$ respectively.*

3 Strong local minimisers and the stability of spherical harmonic twists

Recall that for n even the spherical twist $u(x) = \mathbf{Q}(|x|)x|x|^{-1}$ with the twist path $\mathbf{Q} = \mathbf{Q}(r; m) = \exp(\zeta \mathcal{M}(r)\mathbf{H})$ where $\zeta = 2\pi m + \eta$, $m \in \mathbb{Z}$ and $\mathbf{H} = \text{diag}(\mathbf{J}, \dots, \mathbf{J})$ constitutes a solution to (17), i.e., is a spherical harmonic twist in \mathcal{S}_φ^2 . Here the boundary map is given by (10) where $\mathbf{R} = \text{diag}(\mathcal{R}[\eta], \dots, \mathcal{R}[\eta])$ and $\eta \in [-\pi, \pi)$. Now referring to the description of the second variation of the Dirichlet energy \mathbb{F} in Appendix 1 we have that \mathbb{F} has a positive second variation at u iff,

$$\frac{d^2}{d\varepsilon^2} \mathbb{F}[\Pi(u + \varepsilon\phi)] \Big|_{\varepsilon=0} = 2 \int_{\mathbb{X}^n} \left(|\nabla \hat{\phi}|^2 - |\nabla u|^2 |\hat{\phi}|^2 \right) dx > 0 \quad (41)$$

for all non-zero $\phi \in \mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ with $\hat{\phi} = \phi - \langle u, \phi \rangle u$. Thus here we are after conditions to ensure that the integral on the right in (41) is strictly positive for all non-zero $\hat{\phi} \in \mathcal{W}_0^{1,2}(\mathbb{X}, \mathbb{R}^n)$ satisfying $\langle \hat{\phi}, u \rangle = 0$ almost everywhere in \mathbb{X}^n . When u is the spherical harmonic twist given above, by a basic calculation, the second variation (41) can be explicitly written as

$$\begin{aligned} \frac{d^2}{d\varepsilon^2} \mathbb{F}[\Pi(u + \varepsilon\phi)] \Big|_{\varepsilon=0} &= \frac{d^2}{d\varepsilon^2} \int_{\mathbb{X}} |\nabla[\Pi(u + \varepsilon\phi)]|^2 \Big|_{\varepsilon=0} \\ &= 2 \int_{\mathbb{X}^n} \left(|\nabla \hat{\phi}|^2 - \left[\frac{n-1}{r^2} + \dot{\mathcal{M}}(r)^2 |\zeta|^2 \right] |\hat{\phi}|^2 \right) dx. \end{aligned} \quad (42)$$

In this section we show that by suitably bounding $\zeta = 2\pi m + \eta$ the energy at the spherical harmonic twist u will have a positive second variation and with a further refined analysis that it furnishes a strong local minimiser of the energy.

Lemma 2 *Let $n \geq 3$ and $\mathbf{Q} = \mathbf{Q}(r)$ in $\mathcal{C}([a, b], \mathbf{SO}(n))$ be an arbitrary twist path. Then the inequality*

$$\int_{\mathbb{X}^n} \left(|\nabla \phi|^2 - (n-1) \frac{|\phi|^2}{|x|^2} \right) dx \geq \frac{(n-4)^2}{4} \int_{\mathbb{X}^n} \frac{|\phi|^2}{|x|^2} dx, \quad (43)$$

holds for all $\phi \in \mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$ with $\langle \phi, \mathbf{Q}(|x|)x|x|^{-1} \rangle = 0$ everywhere in \mathbb{X}^n .

Proof Pick $\phi \in \mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$ satisfying the pointwise orthogonality condition $\langle \phi, \mathbf{Q}(r)\theta \rangle = 0$ in \mathbb{X}^n . Using spherical polar coordinates we can then write

$$\begin{aligned} J[\phi; \mathbf{Q}] &:= \int_{\mathbb{X}^n} \left(|\nabla \phi|^2 - (n-1) \frac{|\phi|^2}{|x|^2} \right) dx \\ &= \int_a^b \int_{\{|\omega|=r\}} \left(\left| \frac{\partial \phi}{\partial r} \right|^2 + |\nabla_\omega \phi|^2 - (n-1) \frac{|\phi|^2}{r^2} \right) dr d\mathcal{H}^{n-1}(\omega) \\ &= \int_{\mathbb{X}^n} \left| \frac{\partial \phi}{\partial r} \right|^2 dx + \int_a^b \int_{\{|\theta|=1\}} \frac{1}{r^2} (|\nabla_\theta \phi_r|^2 - (n-1)|\phi_r|^2) d\mathcal{H}^{n-1}(\theta) r^{n-1} dr \\ &= J_1 + J_2, \end{aligned} \quad (44)$$

where on the third line we have set $\phi_r(\theta) = \phi(r\theta)$ for $a \leq r \leq b$ and $|\theta| = 1$. The idea now is to bound each of the two terms J_1 and J_2 separately. We first note that the quadratic form H given by the spherical integral

$$H(f, f) = \int_{\mathbb{S}^{n-1}} (|\nabla_\theta f|^2 - (n-1)|f|^2) d\mathcal{H}^{n-1}(\theta), \quad (45)$$

for $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1}, \mathbb{R}^n)$ satisfying $\langle f, \theta \rangle = 0$ represents the second variation of the Dirichlet energy on the sphere evaluated at the identity map $\varphi : \theta \mapsto \theta$. Hence upon invoking well known results on the spectrum of the corresponding Jacobi operator (cf., e.g., [21, 26]) we have

$$\int_{\mathbb{S}^{n-1}} |\nabla_\theta f|^2 d\mathcal{H}^{n-1}(\theta) \geq 2 \int_{\mathbb{S}^{n-1}} |f|^2 d\mathcal{H}^{n-1}(\theta), \quad (46)$$

for all $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1}, \mathbb{R}^n)$ satisfying $\langle f, \theta \rangle = 0$. Now setting $f(r\theta) = \mathbf{Q}'(r)\phi(r\theta)$ it is easily seen that $\langle f, \theta \rangle = \langle \phi, \mathbf{Q}\theta \rangle = 0$ for all $r \in [a, b]$. Hence, in view of the $\mathbf{SO}(n)$ -invariance of the norms in (46), using a basic density argument, it follows that,

$$\int_{\mathbb{S}^{n-1}} |\nabla_{\theta} \phi(r\theta)|^2 d\mathcal{H}^{n-1}(\theta) \geq 2 \int_{\mathbb{S}^{n-1}} |\phi(r\theta)|^2 d\mathcal{H}^{n-1}(\theta)$$

for all $r \in [a, b]$ and so in the notation of (45) we have

$$H(\phi_r, \phi_r) = \int_{\mathbb{S}^{n-1}} (|\nabla_{\theta} \phi_r|^2 - (n-1)|\phi_r|^2) d\mathcal{H}^{n-1}(\theta) \geq -(n-3) \int_{\mathbb{S}^{n-1}} |\phi_r|^2 d\mathcal{H}^{n-1}(\theta). \quad (47)$$

Referring to (44), upon substitution and a rescaling, this therefore leads to the lower bound on J_2 as,

$$\begin{aligned} J_2 &= \int_a^b \int_{\mathbb{S}^{n-1}} \frac{1}{r^2} (|\nabla_{\theta} \phi_r|^2 - (n-1)|\phi_r|^2) d\mathcal{H}^{n-1}(\theta) r^{n-1} dr \\ &\geq -(n-3) \int_a^b \int_{\mathbb{S}^{n-1}} |\phi_r|^2 d\mathcal{H}^{n-1}(\theta) r^{n-3} dr \\ &\geq -(n-3) \int_{\mathbb{X}^n} \frac{|\phi|^2}{|x|^2} dx. \end{aligned} \quad (48)$$

We next give a lower bound on J_1 by using a Hardy type inequality. Indeed again using polar coordinates and an integration by parts we have

$$\begin{aligned} \int_{\mathbb{X}^n} \frac{|\phi|^2}{r^2} dx &= \frac{1}{n-2} \int_{\mathbb{S}^{n-1}} \int_a^b |\phi_r|^2 \frac{d(r^{n-2})}{dr} dr d\mathcal{H}^{n-1}(\theta) \\ &= -\frac{2}{n-2} \int_{\mathbb{S}^{n-1}} \int_a^b \langle \phi_r, \partial \phi_r / \partial r \rangle r^{n-2} dr d\mathcal{H}^{n-1}(\theta). \end{aligned} \quad (49)$$

Hence by an application of Young's inequality this gives

$$\int_{\mathbb{X}^n} \frac{|\phi|^2}{r^2} dx \leq \frac{1}{2} \int_{\mathbb{X}^n} \frac{|\phi|^2}{r^2} dx + \frac{2}{(n-2)^2} \int_{\mathbb{X}^n} \left| \frac{\partial \phi}{\partial r} \right|^2 dx, \quad (50)$$

and subsequently by rearranging terms it follows that

$$\int_{\mathbb{X}^n} \frac{|\phi|^2}{r^2} dx \leq \frac{4}{(n-2)^2} \int_{\mathbb{X}^n} \left| \frac{\partial \phi}{\partial r} \right|^2 dx = \frac{4}{(n-2)^2} \times J_1. \quad (51)$$

Note that equality in Young's inequality occurs only if $(n-2)\phi = 2r\partial\phi/\partial r$ which then implies $\phi \equiv 0$. Thus, summarising, by combining the above bounds, namely, (48) and (51) we obtain

$$\begin{aligned}
J[\phi; \mathbf{Q}] &= \int_{\mathbb{X}^n} \left(|\nabla \phi|^2 - \frac{n-1}{|x|^2} |\phi|^2 \right) dx \geq \left(\frac{(n-2)^2}{4} - (n-3) \right) \int_{\mathbb{X}^n} \frac{|\phi|^2}{|x|^2} dx \\
&\geq \frac{(n-4)^2}{4} \int_{\mathbb{X}^n} \frac{|\phi|^2}{|x|^2} dx,
\end{aligned} \tag{52}$$

which is the required conclusion. \square

We can now prove one of the main results of the paper. Note that the bound (53) on ζ is stronger than the bound $|\zeta| < \pi$ which is the condition required by Proposition 3. Note also that for the cases $n \geq 3$ odd the conclusion of the theorem still applies (of course with $|n-4|$ in (53) for $n=3$) to the spherical harmonic twist $u = \Pi(x) = x|x|^{-1}$ (corresponding to $\zeta = 0$). However as in odd dimensions radial projections are the only spherical harmonic twists and are indeed globally minimising in $\mathcal{S}_\phi(\mathbb{X})$ below for definiteness we confine only to the remaining cases, i.e., $n \geq 4$ even.

Theorem 2 *Let $n \geq 4$ be even and $u = \mathbf{Q}(r)x|x|^{-1}$ be a spherical harmonic twist. Write $\mathbf{Q}(r) = \exp(\mathcal{G}(r)\mathbf{H}) = \exp(\zeta \mathcal{M}(r)\mathbf{H})$ where $\mathbf{H} = \text{diag}(\mathbf{J}, \dots, \mathbf{J})$ and $\zeta \in \mathbb{R}$ and suppose that*

$$|\zeta| = \|\mathcal{G}\|_{L^\infty(a,b)} \leq \frac{(n-4)}{2(n-2)}(1 - (a/b)^{n-2}). \tag{53}$$

Then the second variation of the Dirichlet energy at the spherical harmonic twist u is positive, that is,

$$\int_{\mathbb{X}^n} (|\nabla \phi|^2 - |\nabla u|^2 |\phi|^2) dx > 0, \tag{54}$$

for all non-zero $\phi \in \mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ satisfying $\langle \phi, \mathbf{Q}(r)x|x|^{-1} \rangle = 0$ a.e. in \mathbb{X}^n .

Proof We first use a density argument to extend the conclusion of Lemma 2 to all $\phi \in \mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ satisfying $\langle \phi, \mathbf{Q}\theta \rangle = 0$ a.e. in \mathbb{X}^n where \mathbf{Q} is as given above. Towards this end fix ϕ as described and pick $(g_j : j \geq 1)$ in $\mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$ so that $g_j \rightarrow \phi$ in $\mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ and put $\phi_j(x) = g_j(x) - \langle g_j(x), \mathbf{Q}(r)\theta \rangle \mathbf{Q}(r)\theta$ for $x \in \mathbb{X}^n$.

Then it is easily seen that $\phi_j \in \mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$, $\langle \phi_j, \mathbf{Q}\theta \rangle = 0$ in \mathbb{X}^n while (ϕ_j) is bounded in $\mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$. As $g_j \rightarrow \phi$, $\langle g_j, \mathbf{Q}\theta \rangle \rightarrow 0$ a.e. in \mathbb{X} this gives $\phi_j \rightarrow \phi$ a.e. in \mathbb{X} . Thus by passing to a subsequence if necessary $\phi_j \rightarrow \phi$ in $\mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ and so by an application of Mazur's Lemma a convex combination of ϕ_j s (still lying in $\mathcal{C}_0^\infty(\mathbb{X}, \mathbb{R}^n)$ and satisfying the orthogonality condition) converges to ϕ in $\mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$. Finally as (43) holds for this approximating sequence built from (ϕ_j) , by passing to the limit $j \nearrow \infty$, it follows that the same must be true for ϕ . Moving on therefore we can write

$$\begin{aligned}
\text{LHS (54)} &= \int_{\mathbb{X}^n} \left(|\nabla \phi|^2 - \left[\frac{n-1}{r^2} + \mathcal{N}(r)^2 |\zeta|^2 \right] |\phi|^2 \right) dx \\
&\geq \int_{\mathbb{X}^n} \left(\frac{(n-4)^2}{4r^2} |\phi|^2 - \mathcal{N}(r)^2 |\zeta|^2 |\phi|^2 \right) dx \\
&> \left[\frac{(n-4)^2}{4} - \frac{a^{4-2n}(n-2)^2}{(b^{2-n} - a^{2-n})^2} |\zeta|^2 \right] \int_{\mathbb{X}^n} |\phi|^2 \frac{dx}{|x|^2} \geq 0,
\end{aligned} \tag{55}$$

where in the last inequality we have used the assumed bound (53) while in the penultimate inequality we have used $\phi \not\equiv 0$ and the formulation of \mathcal{N} .⁶ \square

We now move on to establishing the local minimality in energy space and for natural choices of metrics from the earlier results on the positivity of second variations. The first statement below is functional analytic in nature and uses a classical technique based on the Taylor expansion of the energy functional in the vicinity of a stationary map adapted to the manifold-valued context. The twice Fréchet differentiability assumption on the energy here restricts the scope of the result mainly to energies of curves and twist paths (see Sect. 2) with the energy integral having a strictly quadratic dependence on the gradient argument. Next we turn to the proof of the main theorem and use a different and more direct expansion of the energy integral by invoking the so-called Weierstrass excess function and the sufficiency theorems for strong local minimisers as discussed in [27, 28].

For the sake of the following theorem $M \subset \mathbb{R}^m$ is taken a compact smooth manifold with a tubular neighbourhood $U \subset \mathbb{R}^m$ and nearest point projection $\Pi_M : U \rightarrow M$ which is both well-defined and smooth. For $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain, $\varphi \in \mathcal{C}^1(\partial\Omega, M)$ a fixed boundary map and given $\mathbb{F} = \mathbb{F}[v]$ on $\mathcal{M} := \mathcal{W}_\varphi^{1,2}(\Omega, M) = \{v \in \mathcal{W}^{1,2}(\Omega, M) : v = \varphi \text{ on } \partial\Omega\}$ we set

$$I[v] = \mathbb{F}[\Pi_M(v)] + \|\Pi_M(v) - v\|_{\mathcal{W}^{1,2}}^2, \quad v \in \widehat{\mathcal{M}} := \mathcal{W}_\varphi^{1,2}(\Omega, U) \supset \mathcal{M}. \tag{56}$$

The second term of I in (56) is quadratic and penalises the deviation of v from its pointwise projection $\Pi_M(v)$. Thus $I[v] = \mathbb{F}[v]$ if $v \in \mathcal{M}$. We assume that I agrees with a twice continuously Fréchet differentiable functional (still denoted I) in an open neighbourhood \mathcal{O} of a stationary map $u \in \mathcal{M}$ (see below) in $\mathcal{W}_\varphi^{1,2}(\Omega, \mathbb{R}^m)$. Note the inclusions $\mathcal{W}_\varphi^{1,2}(\Omega, \mathbb{R}^m) \supset \widehat{\mathcal{M}} \supset \mathcal{M}$. See also Remark 2 for related and further discussion.

Theorem 3 *Let $M, \Pi_M, \mathbb{F}, I, \mathcal{M}$ and $\widehat{\mathcal{M}}$ be as above. Assume that $u \in \mathcal{M}$ is of class \mathcal{C}^1 and is a stationary map of the energy \mathbb{F} on \mathcal{M} , that is, for every $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^m)$, we have*

⁶ Note that since the upper bound in (53) is less than π and $\zeta = 2\pi m + \eta$ with $\eta \in [-\pi, \pi)$, it must be that $m = 0$. Also if the inequality in (53) is strict then one can formally improve (54) to $\delta^2 \mathbb{F}[u](\phi, \phi) \geq \gamma \|\phi\|_{L^2(\mathbb{X})}^2$ with $\gamma > 0$. We strengthen (54) shortly using only (53).

$$\left. \frac{d}{d\varepsilon} \mathbb{F}[\Pi_M(u + \varepsilon\phi)] \right|_{\varepsilon=0} = 0. \quad (57)$$

Furthermore assume that there exists $\gamma > 0$ such that

$$\left. \frac{d^2}{d\varepsilon^2} \mathbb{F}[\Pi_M(u + \varepsilon\phi)] \right|_{\varepsilon=0} \geq 2\gamma \|D\Pi_M(u)\phi\|_{\mathcal{H}^{1,2}}^2. \quad (58)$$

Then there exist $\varepsilon, \sigma > 0$ such that for every $v \in \mathcal{M}$ satisfying $\|v - u\|_{\mathcal{H}^{1,2}} < \varepsilon$ we have $\mathbb{F}[v] - \mathbb{F}[u] \geq \sigma \|v - u\|_{\mathcal{H}^{1,2}}^2$.

Proof By choosing U smaller if necessary we assume that the derivatives of Π_M are uniformly bounded on U and so in particular $v \in \widehat{\mathcal{M}} \implies \Pi_M(v) \in \mathcal{M}$.

Consider now the energy $I = I[v]$ for $v \in \mathcal{O}$ as above. Then as I is twice continuously Fréchet differentiable, to justify the assertion, it is enough to show that its first derivative vanishes at u while its second derivative is positive at u . Towards this end pick $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{R}^m)$ and for $\varepsilon \in \mathbb{R}$ put $u_\varepsilon = u + \varepsilon\phi$. Then for ε sufficiently small, u_ε takes values in U and thus $\Pi_M(u_\varepsilon)$ is well-defined, lies in \mathcal{M} and agrees with u when $\varepsilon = 0$. Moreover $\Pi_M(u_\varepsilon) = u + \varepsilon D\Pi_M(u)\phi + o(\varepsilon)$. We compute the first derivative of I at u and verify that it indeed vanishes:

$$\begin{aligned} DI[u](\phi) &= \left. \frac{d}{d\varepsilon} I[u_\varepsilon] \right|_{\varepsilon=0} = \left. \frac{d}{d\varepsilon} \mathbb{F}[\Pi_M(u_\varepsilon)] \right|_{\varepsilon=0} + \left. \frac{d}{d\varepsilon} \|u_\varepsilon - \Pi_M(u_\varepsilon)\|_{\mathcal{H}^{1,2}}^2 \right|_{\varepsilon=0} \\ &= \left. \frac{d}{d\varepsilon} \|\varepsilon\phi - \varepsilon D\Pi_M(u)\phi + o(\varepsilon)\|_{\mathcal{H}^{1,2}}^2 \right|_{\varepsilon=0} = 0, \end{aligned} \quad (59)$$

by virtue of u being a stationary map [cf. (57)] and the regularising energy term in (56) being quadratic. Similarly for the second derivative of I at u we can compute:

$$\begin{aligned} D^2 I[u](\phi, \phi) &= \left. \frac{d^2}{d\varepsilon^2} I[u_\varepsilon] \right|_{\varepsilon=0} = \left. \frac{d^2}{d\varepsilon^2} \mathbb{F}[\Pi_M(u_\varepsilon)] \right|_{\varepsilon=0} + \left. \frac{d^2}{d\varepsilon^2} \|u_\varepsilon - \Pi_M(u_\varepsilon)\|_{\mathcal{H}^{1,2}}^2 \right|_{\varepsilon=0} \\ &\geq 2\gamma \|D\Pi_M(u)\phi\|_{\mathcal{H}^{1,2}}^2 + 2\|\phi - D\Pi_M(u)\phi\|_{\mathcal{H}^{1,2}}^2 \\ &\geq \frac{2\gamma \|\phi\|_{\mathcal{H}^{1,2}}^2}{\gamma + 1}, \end{aligned} \quad (60)$$

where we have used (58). Now for $I[v]$ with $v \in \mathcal{O}$, using the twice continuous Fréchet differentiability of I in \mathcal{O} and by writing the Taylor expansion of I in the vicinity of u in \mathcal{O} , we have

$$\begin{aligned}
I[v] &= I[u] + DI[u][v - u] + \frac{1}{2}D^2I[u][v - u, v - u] + o(\|v - u\|_{\mathcal{W}^{1,2}}^2) \\
&= I[u] + \frac{1}{2}D^2I[u][v - u, v - u] + o(\|v - u\|_{\mathcal{W}^{1,2}}^2) \\
&\geq I[u] + \frac{\gamma/2}{\gamma + 1}\|v - u\|_{\mathcal{W}^{1,2}}^2,
\end{aligned} \tag{61}$$

where we have used (60) and density to deduce the last inequality subject to $\|v - u\|_{\mathcal{W}^{1,2}}$ being sufficiently small. The conclusion now follows by additionally restricting v to \mathcal{M} . \square

Remark 2 For $\Omega \subset \mathbb{R}$ a bounded interval ($n = 1$) $\mathcal{W}_\phi^{1,2}(\Omega, U) \subset \mathcal{W}_\phi^{1,2}(\Omega, \mathbb{R}^m)$ is open. Thus it is enough for I in (56) to be twice continuously Fréchet differentiable in an open neighbourhood \mathcal{O} of u in $\mathcal{W}_\phi^{1,2}(\Omega, U)$ and no extension of I is needed. The twice continuous Fréchet differentiability here is well studied and classical. In particular the \mathbb{G} -energies in Sect. 2 satisfy this condition (cf. [9, 23] for more).

More generally for an energy integral $\mathbb{F} = \mathbb{F}[v]$ with integrand $F = F(x, v, \nabla v)$ one can extend I in (56) to $\mathcal{O} = \mathcal{W}_\phi^{1,2}(\Omega, \mathbb{R}^m)$ by considering the integrand $f(x, v, \nabla v) = \psi(v)[F(x, \Pi_M(v), \nabla \Pi_M(v)) + |\nabla \Pi_M(v) - \nabla v|^2]$ where $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^m)$ verifies $0 \leq \psi \leq 1$, $\psi \equiv 1$ on U and $\psi \equiv 0$ outside V . Here $U \subset \subset V \subset \subset \mathbb{R}^m$ are sufficiently small tubular neighbourhoods of M in \mathbb{R}^m .

One can also pass in Theorem 3 to weaker metrics for local minimisers, e.g., L^1 , subject to the energy satisfying suitable coercivity and bounds (see [29] and Remark 3 below for more). Again these are satisfied by the \mathbb{G} -energies in Sect. 2.

Remark 3 The most natural choice of metric for considering local minimisers of the Dirichlet energy $\mathbb{F}[u]$ over $\mathcal{X}_\phi^2(\mathbb{X}^n)$ is of course the $\mathcal{W}^{1,2}$ -metric. Using an indirect argument we can show however that here any $\mathcal{W}^{1,2}$ -local minimiser of the energy is indeed an L^1 -local minimiser. More specifically that if for a given $u \in \mathcal{W}^{1,2}(\mathbb{X}^n, \mathbb{S}^{n-1})$ there exists $\delta > 0$ such that for every $v \in \mathcal{W}^{1,2}(\mathbb{X}^n, \mathbb{S}^{n-1})$ with $v = u$ on $\partial\mathbb{X}^n$ satisfying $\|u - v\|_{\mathcal{W}^{1,2}} < \delta$ we have $\mathbb{F}[u] \leq \mathbb{F}[v]$ then we indeed have the stronger conclusion that there exists $\varepsilon > 0$ such that for every $v \in \mathcal{W}^{1,2}(\mathbb{X}^n, \mathbb{S}^{n-1})$ with $v = u$ on $\partial\mathbb{X}^n$ satisfying $\|v - u\|_{L^1} < \varepsilon$ we have the energy inequality $\mathbb{F}[u] \leq \mathbb{F}[v]$.

To justify this assume for the sake of a contradiction that the assertion is *false*. Then there exists $v_j \in \mathcal{W}^{1,2}(\mathbb{X}^n, \mathbb{S}^{n-1})$ with $v_j = u$ on $\partial\mathbb{X}^n$ and $v_j \rightarrow u$ in L^1 such that $\mathbb{F}[v_j] < \mathbb{F}[u]$. This in particular results in (v_j) being bounded in $\mathcal{W}^{1,2}$ and so in view of the strong convergence in L^1 , upon passing to a subsequence if necessary, $v_j \rightarrow u$ in $\mathcal{W}^{1,2}$. By combining this weak convergence with the convergence of the energies $\mathbb{F}[v_j] \rightarrow \mathbb{F}[u]$ (note that $\mathbb{F}[u] \leq \liminf \mathbb{F}[v_j]$ and $\mathbb{F}[v_j] < \mathbb{F}[u]$) and writing the basic identity

$$\int_{\mathbb{X}^n} |\nabla v_j - \nabla u|^2 dx = \int_{\mathbb{X}^n} (|\nabla v_j|^2 - |\nabla u|^2) dx + \int_{\mathbb{X}^n} 2\langle \nabla u, \nabla(u - v_j) \rangle dx,$$

it finally follows that $\|v_j - u\|_{\mathcal{W}^{1,2}} \rightarrow 0$ as $j \nearrow \infty$ which along with $\mathbb{F}[v_j] < \mathbb{F}[u]$ contradicts u being a local minimiser with respect to the $\mathcal{W}^{1,2}$ -metric. Thus u is a local minimiser with respect to the L^1 -metric as claimed and the proof is complete. (For more on this see [27, 28, 30].)

Proof of the Main Theorem We justify each of the two parts in the conclusion of the theorem separately.

Part 1. Consider the case $n \geq 3$ odd. Here the only spherical harmonic twists are the radial projections $u \equiv x|x|^{-1}$ corresponding to $\mathbf{Q}(r) \equiv \mathbf{I}_n$ (cf. [31]). Thus in particular $\mathbf{R} = \mathbf{I}_n$ in (10). We now show that u is the global minimiser of the Dirichlet energy \mathbb{F} over $\mathcal{S}_\varphi^2(\mathbb{X}^n)$ with this choice of φ (note that this part of the statement is true regardless of n being even or odd). Indeed if this were not the case then there would be a map $v \in \mathcal{S}_\varphi^2(\mathbb{X}^n)$ with $\mathbb{F}[v; \mathbb{X}^n] < \mathbb{F}[u; \mathbb{X}^n]$. Extending v to the ball \mathbb{B}_b by setting $v \equiv u$ in \mathbb{B}_a we arrive at a contradiction to the minimality of $u = x|x|^{-1}$ in $\mathcal{S}_\varphi^2(\mathbb{B}_b^n)$ (cf. [20]) as then

$$\mathbb{F}[v; \mathbb{B}_b^n] = \mathbb{F}[u; \mathbb{B}_a^n] + \mathbb{F}[v; \mathbb{X}^n] < \mathbb{F}[u; \mathbb{B}_a^n] + \mathbb{F}[u; \mathbb{X}^n] = \mathbb{F}[u; \mathbb{B}_b^n]. \quad (62)$$

Part 2. We now consider the case $n \geq 4$ even. Indeed for the sake of clarity of presentation and convenience of the reader we split this part into three steps.

Step 1. Here we use Theorem 2 to show that subject to the conditions stated in the main theorem the second variation of the energy at the spherical harmonic twist u satisfies the stronger inequality,

$$\frac{d^2}{d\varepsilon^2} \mathbb{F}[\Pi(u + \varepsilon\phi)] \Big|_{\varepsilon=0} = \frac{d^2}{d\varepsilon^2} \int_{\mathbb{X}^n} |\nabla(\Pi(u + \varepsilon\phi))|^2 dx \Big|_{\varepsilon=0} \geq 2\gamma \|\hat{\phi}\|_{\mathcal{W}^{1,2}}^2, \quad (63)$$

for all $\phi \in \mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ where $\hat{\phi} = \phi - \langle u, \phi \rangle u$, hence, $\langle \hat{\phi}, u \rangle = 0$ almost everywhere in \mathbb{X}^n and for some $\gamma > 0$. Now recall that the second variation of the energy at u is given by the quadratic form

$$\frac{d^2}{d\varepsilon^2} \int_{\mathbb{X}^n} |\nabla(\Pi(u + \varepsilon\phi))|^2 dx \Big|_{\varepsilon=0} = 2 \int_{\mathbb{X}^n} (|\nabla \hat{\phi}|^2 - |\nabla u|^2 |\hat{\phi}|^2) dx := 2\mathcal{Q}[\hat{\phi}], \quad (64)$$

and this by Theorem 2 is strictly positive if $\hat{\phi}$ is non-zero. Now in order to prove (63) we argue by contradiction and invoke the quadratic nature of \mathcal{Q} as expressing the second variation in (64). Indeed, if the assertion were false, there would exist a sequence $\hat{\phi}_k \in \mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ satisfying $\langle \hat{\phi}_k, u \rangle = 0$ almost everywhere in \mathbb{X}^n such that

$$\int_{\mathbb{X}^n} |\nabla \hat{\phi}_k|^2 dx = 1, \quad 0 < \mathcal{Q}[\hat{\phi}_k] < \frac{1}{k}. \quad (65)$$

As $\|\hat{\phi}_k\|_{\mathcal{W}^{1,2}}$ is bounded (by an application of the Poincaré inequality) there exists $\phi \in \mathcal{W}_0^{1,2}(\mathbb{X}^n, \mathbb{R}^n)$ such that (possibly on a subsequence) $\hat{\phi}_k \rightharpoonup \phi$ in $\mathcal{W}_0^{1,2}$. Furthermore by the Rellich-Kondrachov compactness theorem this gives the L^2 -strong and pointwise convergence (of a further subsequence if necessary) $\hat{\phi}_k \rightarrow \phi$ and so $\langle \phi, u \rangle = 0$ almost everywhere in \mathbb{X}^n . Now with these at hand we can write

$$\begin{aligned} \left| 1 - \int_{\mathbb{X}^n} |\nabla u|^2 |\phi|^2 dx \right| &= \left| \int_{\mathbb{X}^n} (|\nabla \hat{\phi}_k|^2 - |\nabla u|^2 |\phi|^2) dx \right| \\ &\leq \left| \int_{\mathbb{X}^n} |\nabla u|^2 (|\hat{\phi}_k|^2 - |\phi|^2) dx \right| + \left| \int_{\mathbb{X}^n} (|\nabla \hat{\phi}_k|^2 - |\nabla u|^2 |\hat{\phi}_k|^2) dx \right| \quad (66) \\ &\leq \sup_{\mathbb{X}^n} |\nabla u|^2 \left| \int_{\mathbb{X}^n} (|\hat{\phi}_k|^2 - |\phi|^2) dx \right| + \mathcal{Q}[\hat{\phi}_k]. \end{aligned}$$

It therefore follows upon letting $k \nearrow \infty$ and noting the second inequality in (65) that $\|\phi \nabla u\|_{L^2} = 1$ and so in particular $\phi \not\equiv 0$. Now by invoking the sequential weak lower semicontinuity of $\mathcal{Q}[\phi]$ on $\mathcal{W}_0^{1,2}$ together with Theorem 2, it follows that

$$0 < \mathcal{Q}[\phi] \leq \liminf_{k \nearrow \infty} \mathcal{Q}[\hat{\phi}_k] \leq \liminf_{k \nearrow \infty} \frac{1}{k} = 0, \quad (67)$$

with the *strict* inequality being a result of ϕ being non-zero. The contradiction is thus reached and so we conclude that the inequality (63) must be true.

Step 2. Let us now proceed by introducing an unconstrained energy functional by way of setting

$$\begin{aligned} I[v] &= \mathbb{F}[\Pi(v)] + \|\Pi(v) - v\|_{\mathcal{W}^{1,2}}^2 \\ &= \int_{\mathbb{X}^n} |\nabla \Pi(v)|^2 + \int_{\mathbb{X}^n} |\nabla \Pi(v) - \nabla v|^2. \quad (68) \end{aligned}$$

As before Π is the radial projection and it is assumed hereafter that the competing maps v take values in a fixed open neighbourhood of the unit sphere in \mathbb{R}^n away from the origin (see below). Clearly if $|v| = 1$ almost everywhere in \mathbb{X}^n , that is, if v takes values on the unit sphere, then $\Pi(v) \equiv v$ and therefore $I[v] = \mathbb{F}[v]$.

Now take $v = u + \varepsilon \phi$ where the map u is as in the statement of the theorem, $\phi \in \mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$ and $\varepsilon \in \mathbb{R}$ is sufficiently small. Then it is easily seen that $\Pi(v) = \Pi(u + \varepsilon \phi) = u + \varepsilon D\Pi(u)\phi + o(\varepsilon)$ and $\nabla \Pi(v) = \nabla u + \varepsilon \nabla(D\Pi(u)\phi) + o(\varepsilon)$. Hence calculating the first and second variations of I at u , by a straightforward calculations it is seen that $\|\Pi(v) - v\|_{\mathcal{W}^{1,2}}^2 = \varepsilon^2 \|\nabla(D\Pi(u)\phi) - \nabla \phi\|_{L^2}^2 + o(\varepsilon^2)$, and so

$$\left. \frac{d}{d\varepsilon} I[u + \varepsilon \phi] \right|_{\varepsilon=0} = \int_{\mathbb{X}} \langle \nabla u, \nabla(D\Pi(u)\phi) \rangle = \int_{\mathbb{X}} \langle \nabla u, \nabla[(\mathbf{I}_n - u \otimes u)\phi] \rangle = 0, \quad (69)$$

as a result of u being a spherical harmonic twist and in a similar way

$$\begin{aligned} \left. \frac{d^2}{d\varepsilon^2} I[u + \varepsilon\phi] \right|_{\varepsilon=0} &= \left. \frac{d^2}{d\varepsilon^2} \int_{\mathbb{X}} |\nabla[\Pi(u + \varepsilon\phi)]|^2 \right|_{\varepsilon=0} + 2 \int_{\mathbb{X}} |\nabla(D\Pi(u)\phi) - \nabla\phi|^2 \\ &= 2 \int_{\mathbb{X}} |\nabla\hat{\phi}|^2 - |\nabla u|^2 |\hat{\phi}|^2 + |\nabla(D\Pi(u)\phi) - \nabla\phi|^2 \\ &\geq 2\gamma \|\hat{\phi}\|_{\mathcal{H}^{1,2}}^2 + 2\|\hat{\phi} - \phi\|_{\mathcal{H}^{1,2}}^2 \geq \gamma \|\phi\|_{\mathcal{H}^{1,2}}^2. \end{aligned} \quad (70)$$

(Here we are assuming without loss of generality that $\gamma \in]0, 1[$ as a consequence of which $\gamma = \min(1, \gamma)$.) Now referring to the formulation of the unconstrained energy (68), by direct differentiation of the radial projection and substitution, we can write,

$$\begin{aligned} I[v] &= \int_{\mathbb{X}} |D\Pi(v)\nabla v|^2 dx + \int_{\mathbb{X}} |D\Pi(v)\nabla v - \nabla v|^2 dx \\ &= \int_{\mathbb{X}} \left| \left(\mathbf{I}_n - \frac{v}{|v|} \otimes \frac{v}{|v|} \right) \frac{\nabla v}{|v|} \right|^2 dx + \int_{\mathbb{X}} \left| \left((1 - |v|)\mathbf{I}_n - \frac{v}{|v|} \otimes \frac{v}{|v|} \right) \frac{\nabla v}{|v|} \right|^2 dx \\ &= \int_{\mathbb{X}} \left\{ \frac{|\nabla v|^2}{|v|^2} - \frac{|(\nabla v)^t v|^2}{|v|^4} + (1 - |v|)^2 \frac{|\nabla v|^2}{|v|^2} + (2|v| - 1) \frac{|(\nabla v)^t v|^2}{|v|^4} \right\} dx \\ &= \int_{\mathbb{X}} f(x, v, \nabla v) dx, \end{aligned} \quad (71)$$

where referring to the last identity in (71) the integrand f here has the explicit form

$$f(x, v, \nabla v) = (|v|^2 - 2|v| + 2) \frac{|\nabla v|^2}{|v|^2} + 2(|v| - 1) \frac{|(\nabla v)^t v|^2}{|v|^4}. \quad (72)$$

Note that a direct calculation using (68) or from the explicit description (72) and the identity $|v||D\Pi(v)\nabla v|^2 = \langle \nabla v, D\Pi(v)\nabla v \rangle$ gives the lower bound $f(x, v, \nabla v) \geq |\nabla v|^2/2$. Proceeding forward now recall that the Weierstrass excess function associated with an integrand $f = f(x, v, p)$ is given by (cf., e.g., [27] pp. 181)

$$E_f(x, v, p, q) = f(x, v, q) - f(x, v, p) - \langle f_p(x, v, p), (q - p) \rangle. \quad (73)$$

This function is a key player in both the formulation and proof of the sufficiency theorem for strong local minimiser as used in step 3 below. As a matter of fact for the integrand f in (72) and for non-zero v (i.e., $|v| > 0$) and arbitrary $p, q \in \mathbb{R}^{n \times n}$ a straightforward calculation gives

$$\begin{aligned}
E_f(x, v, p, q) &= (|v|^2 - 2|v| + 2) \frac{|q - p|^2}{|v|^2} + 2(|v| - 1) \frac{|(q - p)^t v|^2}{|v|^4} \\
&= |q - p|^2 + 2 \frac{1 - |v|}{|v|^2} |q - p|^2 + 2 \frac{|v| - 1}{|v|^4} |(q^t - p^t)v|^2 \\
&= \frac{1}{2} |q - p|^2 + \frac{1}{2} \left| \left[\mathbf{I}_n - \frac{2}{|v|} \left(\mathbf{I}_n - \frac{v}{|v|} \otimes \frac{v}{|v|} \right) \right] (q - p) \right|^2 \geq \frac{1}{2} |q - p|^2.
\end{aligned} \tag{74}$$

Step 3. We are now in a position to apply the sufficiency theorem for strong local minimisers ([27] Theorem 3.3) and conclude. Towards this end and as shown in step 1 above the stationary harmonic twist u satisfies (69)–(70). Therefore by virtue of the Weierstrass excess function inequality (74) in step 2 it follows from [27] Theorem 3.3 part (1) that $\exists \varepsilon, \sigma > 0$ so that with $U = \mathbb{B}_{3/2} \setminus \overline{\mathbb{B}_{1/2}}$: $\forall v \in \mathcal{W}^{1,2}(\mathbb{X}^n, U)$, $v = u$ on $\partial \mathbb{X}^n$ and $\|v - u\|_{L^\infty} < \varepsilon \implies I[v] - I[u] \geq \sigma \|v - u\|_{\mathcal{W}^{1,2}}^2$. Thus by restricting v further to $\mathcal{S}_\varphi^2(\mathbb{X}^n)$ we have $\mathbb{F}[v] - \mathbb{F}[u] = I[v] - I[u] \geq \sigma \|v - u\|_{\mathcal{W}^{1,2}}^2$.

We can improve the above statement to u being an L^1 -local minimiser of I over $\mathcal{W}^{1,2}(\mathbb{X}^n, U)$ by slightly adjusting the argument in [27] Theorem 3.3 part (2). Indeed pick $\psi \in \mathcal{C}_0^\infty(0, \infty)$ such that $0 \leq \psi \leq 1$ with $\psi \equiv 1$ on $[1/2, 3/2]$ and $\psi \equiv 0$ outside $(1/4, 7/4)$. Replacing the integrand f by $\psi(|v|)f$ does not change $I[v]$ for $v \in \mathcal{W}^{1,2}(\mathbb{X}^n, U)$ but overcomes the indeterminacy of f at $v = 0$. In particular here we still have (69)–(70). Now a close inspection of the proof of [27] Theorem 3.3 part (2) reveals that all that is required of the Weierstrass excess function along the sequence (u_j) in $\mathcal{W}^{1,2}(\mathbb{X}^n, U)$ with $u_j \rightarrow u$ in $\mathcal{W}^{1,2}$ and $I[u_j] - I[u] < 1/j \|u_j - u\|_{\mathcal{W}^{1,2}}^2$ is to satisfy $E_f(x, u_j, \nabla u, \nabla u_j) \geq \alpha \|u_j - u\|_{\mathcal{W}^{1,2}}^2$ which is certainly true here with $\alpha = 1/2$ as a result of (74). It thus follows that $\exists \varepsilon, \sigma > 0$ so that: $\forall v \in \mathcal{W}^{1,2}(\mathbb{X}^n, U)$, $v = u$ on $\partial \mathbb{X}^n$ and $\|v - u\|_{L^1} < \varepsilon \implies I[v] - I[u] \geq \sigma \|v - u\|_{\mathcal{W}^{1,2}}^2$. Again by restricting v further to $\mathcal{S}_\varphi^2(\mathbb{X}^n)$ we have $\mathbb{F}[v] - \mathbb{F}[u] = I[v] - I[u] \geq \sigma \|v - u\|_{\mathcal{W}^{1,2}}^2$. This at once gives the desired conclusion and thus completes the proof. \square

Compliance with ethical standards

Conflict of interest The authors declare that they do not have conflict of interests.

Ethical standard This research complies with ethical standards.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Appendix

This appendix has two distinct aims: firstly, to formalise the calculations of the first and second variations of the \mathbb{F} -energy integral (1) over \mathcal{S}_φ , and secondly, to establish a minimality property for solutions of the Euler–Lagrange system (2) in homotopy classes when $n = 2$.

Formulation of energy variations when $n \geq 2$

The aim here is to give a formal derivation for the first and second variations of the energy (1) with F a twice continuously differentiable function. Let $u \in \mathcal{S}_\varphi$ be of class \mathcal{C}^1 and for $\phi \in \mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$ and $\varepsilon \in \mathbb{R}$ sufficiently small (so that $|u + \varepsilon\phi| \geq 1/2$ in \mathbb{X}^n) put $u_\varepsilon = \Pi(u + \varepsilon\phi) = (u + \varepsilon\phi)/|u + \varepsilon\phi|$. A straightforward calculation firstly gives

$$\left. \frac{d}{d\varepsilon} u_\varepsilon \right|_{\varepsilon=0} = \phi - \langle u, \phi \rangle u, \quad \left. \frac{d^2}{d\varepsilon^2} u_\varepsilon \right|_{\varepsilon=0} = 3\langle u, \phi \rangle^2 u - |\phi|^2 u - 2\langle u, \phi \rangle \phi. \quad (75)$$

Subsequently by referring to (1) we can compute

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \mathbb{F}[u_\varepsilon; \mathbb{X}^n] \right|_{\varepsilon=0} &= 2 \int_{\mathbb{X}^n} F'(|\nabla u|^2) \langle \nabla u_\varepsilon, \nabla \frac{d}{d\varepsilon} u_\varepsilon \rangle dx \Big|_{\varepsilon=0} \\ &= 2 \int_{\mathbb{X}^n} F'(|\nabla u|^2) [\langle \nabla u, \nabla \phi \rangle - |\nabla u|^2 \langle u, \phi \rangle] dx, \end{aligned} \quad (76)$$

that upon assuming u is a stationary map gives (2). Next using $\hat{\phi} = D\Pi(u)\phi = \phi - \langle u, \phi \rangle u$ as before and with the same assumption on u , the second variation at u (recalling the vanishing of the first energy variation at u) can be written as

$$\begin{aligned} \left. \frac{d^2}{d\varepsilon^2} \mathbb{F}[u_\varepsilon; \mathbb{X}^n] \right|_{\varepsilon=0} &= \int_{\mathbb{X}^n} 4F''(|\nabla u_\varepsilon|^2) \langle \nabla u_\varepsilon, \nabla \frac{d}{d\varepsilon} u_\varepsilon \rangle^2 + 2F'(|\nabla u_\varepsilon|^2) \left| \nabla \frac{d}{d\varepsilon} u_\varepsilon \right|^2 \\ &\quad + 2F'(|\nabla u_\varepsilon|^2) \langle \nabla u_\varepsilon, \nabla \frac{d^2}{d\varepsilon^2} u_\varepsilon \rangle dx \Big|_{\varepsilon=0} \\ &= \int_{\mathbb{X}^n} 4F''(|\nabla u|^2) \langle \nabla u, \nabla \hat{\phi} \rangle^2 + 2F'(|\nabla u|^2) |\nabla \hat{\phi}|^2 \\ &\quad + 2F'(|\nabla u|^2) \langle u | \nabla u|^2, \frac{d^2}{d\varepsilon^2} u_\varepsilon \rangle \Big|_{\varepsilon=0} dx \\ &= \int_{\mathbb{X}^n} 4F''(|\nabla u|^2) \langle \nabla u, \nabla \hat{\phi} \rangle^2 + 2F'(|\nabla u|^2) |\nabla \hat{\phi}|^2 \\ &\quad - 2F'(|\nabla u|^2) |\nabla u|^2 |\hat{\phi}|^2 dx. \end{aligned} \quad (77)$$

Energy minimisers in the homotopy classes of $\mathcal{A}_\varphi^p(\mathbb{X})$ when $n = 2$

In this second part of the appendix we show that in the planar case $n = 2$ there is a stronger stability result for the spherical harmonic twists by way of proving that sufficiently smooth solutions of the associated Euler–Lagrange system minimise energy in their respective homotopy classes. To formalise this recall that here we are considering the energy integral

$$\mathbb{F}[u; \mathbb{X}] := \int_{\mathbb{X}} F(|\nabla u|^2) dx, \quad (78)$$

(hereafter $\mathbb{X} = \mathbb{X}^2$) under similar assumptions on the integrand F as before (i.e., F is twice continuously differentiable, is bounded from below – without loss of generality $F(t) \geq 0$ – and $F'(t), F''(t) \geq 0$ for all $t \geq 0$) and over the space

$$\mathcal{A}_\varphi^p(\mathbb{X}) = \{u \in \mathcal{W}^{1,p}(\mathbb{X}, \mathbb{S}^1) : u = \varphi \text{ on } \partial\mathbb{X}\}, \quad p \geq 2. \quad (79)$$

Here $\varphi \in \mathcal{C}^1(\partial\mathbb{X}, \mathbb{S}^1)$ and is the restriction to $\partial\mathbb{X}$ of a map in $\mathcal{C}(\overline{\mathbb{X}}, \mathbb{S}^1)$ (this in particular ensures that $\mathcal{A}_\varphi^p(\mathbb{X})$ is non-empty). We show that with the convexity-monotonicity assumption on F solutions u of the Euler–Lagrange system (2) are energy minimisers in their respective homotopy classes. Now to describe exactly what we mean by homotopy classes first note that any $u \in \mathcal{C}(\overline{\mathbb{X}}, \mathbb{S}^1)$ can be regarded as a map (again called u) in $\mathcal{C}(\mathcal{R}, \mathbb{S}^1)$ with $\mathcal{R} = [a, b] \times [0, 2\pi]$ such that (with a slight abuse of notation) $u(r, 0) = u(r, 2\pi)$ for all $a \leq r \leq b$. Since $\mathcal{R} \subset \mathbb{R}^2$ is simply-connected u has a lifting $f \in \mathcal{C}(\mathcal{R}, \mathbb{R})$ so that $u = \exp(if)$.⁷ Now as the fibre of each point in \mathbb{S}^1 under the covering map $\exp(ix) : \mathbb{R} \rightarrow \mathbb{S}^1$ is discrete and countably infinite (indeed is a translation of the lattice $2\pi\mathbb{Z} \subset \mathbb{R}$) in order to define f uniquely and unambiguously it suffices to fix a base point $\zeta \in \mathbb{R}$ in the fibre of $\varphi(a, 0) \in \mathbb{S}^1$ and then set $f(a, 0) = \zeta$. Now by continuity $f(r, 2\pi) - f(r, 0) = 2\pi k$ for all $a \leq r \leq b$ and some $k \in \mathbb{Z}$ and if $f, g \in \mathcal{C}(\mathcal{R}, \mathbb{R})$ denote the liftings of $u, v \in \mathcal{C}_\varphi(\overline{\mathbb{X}}, \mathbb{S}^1) = \{w \in \mathcal{C}(\overline{\mathbb{X}}, \mathbb{S}^1) : w = \varphi \text{ on } \partial\mathbb{X}\}$ then $g(b, 0) - f(b, 0) = 2\pi m$ for some $m \in \mathbb{Z}$. (Here k is the common value of the Brouwer degrees of either map $\varphi|_{\partial\mathbb{X}_a}$ or $\varphi|_{\partial\mathbb{X}_b}$ that agree in view of $\varphi \in \mathcal{C}(\overline{\mathbb{X}}, \mathbb{S}^1)$ and the single integer m solely characterises the different homotopy classes of $\mathcal{C}_\varphi(\overline{\mathbb{X}}, \mathbb{S}^1)$.) Indeed if u, v are homotopic in $\mathcal{C}_\varphi(\overline{\mathbb{X}}, \mathbb{S}^1)$ (relative to $\partial\mathbb{X}$) then $f \equiv g$ on the vertical part of the boundary $\partial\mathcal{R}_v = \{r = a\} \cup \{r = b\}$ and conversely if $f \equiv g$ on $\partial\mathcal{R}_v$ then u, v are homotopic in $\mathcal{C}_\varphi(\mathbb{X}, \mathbb{S}^1)$ (relative to $\partial\mathbb{X}$). As a matter of fact here a homotopy between u, v is given by $w_t = \exp(ih_t)$ where $h_t = (1-t)f + tg$ and $0 \leq t \leq 1$. Note that $h_t \equiv f \equiv g$ on \mathcal{R}_v for all $0 \leq t \leq 1$ while $h_t(r, 2\pi) - h_t(r, 0) = (1-t)(f(r, 2\pi) - f(r, 0)) + t(g(r, 2\pi) - g(r, 0)) = 2\pi k$ for all $a \leq r \leq b$ and $0 \leq t \leq 1$. The homotopy classes of $\mathcal{C}_\varphi(\overline{\mathbb{X}}, \mathbb{S}^1)$ and $\mathcal{A}_\varphi^p(\mathbb{X})$ ($p > 2$) are thus understood in this sense. The case $p = 2$ is similar but requires a slightly more involved argument (cf. [18, 29, 34]).

⁷ Here we are identifying $u = (u_1, u_2)$ with its complex representation $u = u_1 + iu_2$ in view of the inclusion $\mathbb{S}^1 \subset \mathbb{C}$.

For a discussion of liftings (existence and regularity) in the context of Sobolev spaces see [5] and the references therein. The proof that solutions of the Euler–Lagrange system (80) minimise energy in their respective homotopy classes is now based on the observation that, upon writing $u = \exp(if)$, the system (80) transforms into a single PDE for f (82). As two admissible maps are homotopic iff their liftings as above agree on $\partial\mathcal{R}_v$, the desired energy inequality follows by periodicity, integration by parts and a convexity-monotonicity argument.

Theorem 4 *Let $F = F(t)$ be a twice continuously differentiable, monotone increasing and convex function satisfying $F'(t), F''(t) \geq 0$ for $t \geq 0$. Then if $u \in \mathcal{C}^2(\overline{\mathbb{X}}, \mathbb{S}^1)$ is a solution to the system*

$$\begin{cases} \operatorname{div} [F'(|\nabla u|^2) \nabla u] + F'(|\nabla u|^2) |\nabla u|^2 u = 0 & \text{in } \mathbb{X}, \\ u = \varphi & \text{on } \partial\mathbb{X}, \end{cases} \quad (80)$$

then u is a minimiser of the energy \mathbb{F} (78) in its homotopy class.

Proof Let $u \in \mathcal{C}^2(\overline{\mathbb{X}}, \mathbb{S}^1)$ be a solution to (80) and let f be the lifting of u as described above so that $u = \exp(if)$ in $\mathcal{R} = \{(r, \theta) : a \leq r \leq b, 0 \leq \theta \leq 2\pi\} \subset \mathbb{R}^2$. Then $|\nabla u|^2 = |\nabla f|^2 = (\partial_r f)^2 + 1/r^2 (\partial_\theta f)^2$ and so

$$\begin{aligned} \mathcal{L}[u = \exp(if)] &:= \operatorname{div} [F'(|\nabla u|^2) \nabla u] + F'(|\nabla u|^2) |\nabla u|^2 u \\ &= \operatorname{div} [F'(|\nabla f|^2) (\cos f, \sin f)^\perp \otimes \nabla f] + F'(|\nabla f|^2) |\nabla f|^2 (\cos f, \sin f)^t \\ &= \operatorname{div} [F'(|\nabla f|^2) \nabla f] ((\cos f, \sin f)^\perp)^t \end{aligned} \quad (81)$$

where $(\cos f, \sin f)^\perp = (-\sin f, \cos f)$.⁸ Thus it is seen from the system (80) for u that the corresponding equation for the scalar function f is a single PDE, that is,

$$\mathcal{L}[u] = 0 \iff \operatorname{div} [F'(|\nabla f|^2) \nabla f] = 0, \quad (82)$$

or more explicitly in the (r, θ) variables

$$\partial_r [r F'(|\nabla f|^2) \partial_r f] + \frac{1}{r} \partial_\theta [F'(|\nabla f|^2) \partial_\theta f] = 0. \quad (83)$$

Now pick a competing map v in the same homotopy class as u and with $v \equiv \varphi$ on $\partial\mathbb{X}$. Then the lifting g of v satisfies $g = f$ on $\partial\mathcal{R}_v$. Hence using the convexity and monotonicity of F we can bound the difference in the energies of v and u as

⁸ Note that the divergence operator on the first and second lines above acts on matrix fields and row-wise whilst the divergence operator on the third line is the usual one acting on vector fields.

$$\begin{aligned}
\mathbb{F}[v; \mathbb{X}] - \mathbb{F}[u; \mathbb{X}] &= \int_{\mathbb{X}} (F(|\nabla v|^2) - F(|\nabla u|^2)) dx \\
&= \int_{\mathcal{R}} (F(|\nabla g|^2) - F(|\nabla f|^2)) r dr d\theta \\
&\geq \int_{\mathcal{R}} F'(|\nabla f|^2) (|\nabla g|^2 - |\nabla f|^2) r dr d\theta \\
&\geq \int_{\mathcal{R}} F'(|\nabla f|^2) (|\nabla g|^2 - |\nabla(g-f)|^2 - |\nabla f|^2) r dr d\theta \\
&\geq \int_{\mathcal{R}} 2F'(|\nabla f|^2) (\langle \nabla f, \nabla g \rangle - |\nabla f|^2) r dr d\theta \\
&\geq \int_{\mathcal{R}} 2F'(|\nabla f|^2) \langle \nabla f, \nabla(g-f) \rangle r dr d\theta.
\end{aligned} \tag{84}$$

Note that the third line uses the convexity of F and the fourth line follows from the non-negativity of F' . Now setting $\phi = g - f$ we have $\phi \equiv 0$ on $\partial\mathcal{R}_v$ and $\phi(r, 2\pi) = \phi(r, 0)$ for $a \leq r \leq b$ as a result of $f(r, 2\pi) - f(r, 0) = 2\pi k$, $g(r, 2\pi) - g(r, 0) = 2\pi k$. Thus integrating by parts and using (83) we can write

$$\begin{aligned}
\int_{\mathcal{R}} F'(|\nabla f|^2) \langle \nabla f, \nabla \phi \rangle r dr d\theta &= \int_{\mathcal{R}} F'(|\nabla f|^2) (\partial_r f \partial_r \phi + \frac{1}{r^2} \partial_{\theta} f \partial_{\theta} \phi) r dr d\theta \\
&= \int_{\mathcal{R}} - \left\{ \partial_r (r F' \partial_r f) + \frac{1}{r} \partial_{\theta} (F' \partial_{\theta} f) \right\} \phi dr d\theta \\
&\quad + \int_{\partial\mathcal{R}} (r F' \partial_r f, \frac{1}{r} F' \partial_{\theta} f) \cdot \nu \phi d\mathcal{H}^1 \\
&= \int_a^b \left\{ (F' \partial_{\theta} f) \Big|_{\theta=2\pi} - (F' \partial_{\theta} f) \Big|_{\theta=0} \right\} \frac{\phi}{r} = 0,
\end{aligned}$$

where the last equality follows upon noting that by periodicity and smoothness, $f(r, \theta + 2\pi) - f(r, \theta) = 2k\pi$ and so $\partial_r f(r, 2\pi) = \partial_r f(r, 0)$, $\partial_{\theta} f(r, 2\pi) = \partial_{\theta} f(r, 0)$ and thus $|\nabla f|^2(r, 2\pi) = |\nabla f|^2(r, 0)$. Hence substituting back in (84) gives $\mathbb{F}[v; \mathbb{X}] - \mathbb{F}[u; \mathbb{X}] \geq 0$ and so the conclusion follows. \square

References

1. Alouges, F., and J. Ghidaglia. 1997. Minimizing Oseen-Frank energy for nematic liquid crystals: Algorithms and numerical results. *Annales de l'IHP Physique théorique* 66: 411–447.

2. Baldes, A. 1984. Stability and uniqueness properties of the equator map from a ball into an ellipsoid. *Math. Z.* 185 (4): 505–516.
3. Berger, M. 2007. *A Panoramic view of Riemannian Geometry*. Berlin: Springer.
4. Bedford, S. 2016. Function spaces for liquid crystal theory. *Archive for Rational Mechanics and Analysis* 219 (2): 937–984.
5. Bourgain, J., H. Brezis, and P. Mironescu. 2000. Lifting in Sobolev spaces. *Journal d'analyse Mathématique* 80 (1): 37–86.
6. Bourgain, J.C. 2006. The minimality of the map $x/||x||$ for weighted energy. *Calculus of Variations and Partial Differential* 25 (4): 469–489.
7. Brezis, H., Y. Li, P. Mironescu, and L. Nirenberg. 1999. Degree and Sobolev spaces. *Topological Methods in Nonlinear Analysis* 13 (2): 181–190.
8. Cicalese, M., M. Ruf, and F. Solombrino. 2017. On global and local minimizers of prestrained thin elastic rods. *Topological Methods in Nonlinear Analysis* 56 (4): 115.
9. Cesari, L. 1983. *Optimization theory and applications: Problems with ordinary differential equations, applications of mathematics*, vol. 17. Berlin: Springer.
10. Chavel, I. 2006. *Riemannian Geometry: A Modern Introduction*, vol. 98., Cambridge Studies in Advanced Mathematics: Cambridge University Press.
11. Day, S., and A. Taheri. 2018. Geodesics on $\mathbf{SO}(n)$ and a class of spherically symmetric mappings as solutions to a nonlinear generalised harmonic map problem. *Topological Methods in Nonlinear Analysis* 51 (2): 637–662.
12. Day, S., and A. Taheri. 2017. A class of extremising sphere-valued maps with inherent maximal tori symmetries in $\mathbf{SO}(n)$. *Boundary Value Problems* 2017: 187.
13. Eells, J., and L. Lemaire. 1978. A report on harmonic maps. *Bulletin of the London Mathematical Society* 10: 1–68.
14. Eells, J., and L. Lemaire. 1988. Another report on harmonic maps. *Bulletin of the London Mathematical Society* 20: 385–524.
15. Ericksen, J.L. 1966. Inequalities in liquid crystal theory. *Physics of Fluids* 9 (6): 1205–1207.
16. Giaquinta, M., G. Modica, and J. Soucek. 1998. *Cartesian Currents in the Calculus of Variations*, vol. I & II. Berlin: Springer.
17. Fegan, H.D. 1991. *Introduction to Compact Lie Groups*, vol. 13., Series in Pure Mathematics Singapore: World Scientific.
18. Hang, F.B., and F.H. Lin. 2001. Topology of Sobolev mappings. *Mathematical Research Letters* 8: 321–330.
19. Knapp, A.W. 2002. *Lie Groups beyond an introduction, progress in mathematics*, 2nd ed. Basel: Birkhäuser.
20. Lin, F.H. 1987. A remark on the map $x/|x|$. *CRA Science* 305 (12): 529–531.
21. Mazet, E. 1973. La formule de la variation seconde de l'énergie au voisinage d'une application harmonique. *Journal of Differential Geometry* 8 (2): 279–296.
22. Morrison, G., and A. Taheri. 2018. An infinite scale of incompressible twisting solutions to the nonlinear elliptic system $\mathcal{A}[u; A, B] = \nabla \mathcal{P}$ and the discriminant $\Delta(h, g)$. *Nonlinear Analysis* 173: 209–219.
23. Noll, D. 1993. Second order differentiability of integral functionals on Sobolev spaces and L^2 -spaces. *Journal für die Reine und Angewandte Mathematik* 436: 1–17.
24. Sepanski, M.R. 2007. *Compact Lie Groups, Graduate Texts in Mathematics*, vol. 235. Berlin: Springer.
25. Simon, B. 1995. *Representations of Finite and Compact Groups*, vol. 10., Graduate Studies in Mathematics Providence: American Mathematical Society.
26. Smith, R.T. 1975. The second variation formula for harmonic mappings. *Proceedings of the American Mathematical Society* 47 (1): 229–236.
27. Taheri, A. 2001. Sufficiency theorems for local minimizers of the multiple integrals of the calculus of variations. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 131: 155–184.
28. Taheri, A. 2002. Strong versus weak local minimizers for the perturbed Dirichlet functional. *Calculus of Variations and Partial Differential Equations* 15 (2): 215–235.
29. Taheri, A. 2005. Local minimizers and quasiconvexity—the impact of topology. *Archive for Rational Mechanics and Analysis* 176 (3): 363–414.
30. Taheri, A. 2010. Homotopy classes of self-maps of annuli, generalised twists and spin degree. *Archive for Rational Mechanics and Analysis* 197 (1): 239–270.

31. Taheri, A. 2012. Spherical twists, stationary paths and harmonic maps from generalised annuli into spheres. *NoDEA* 19: 79–95.
32. Taheri, A. 2015. *Function Spaces and Partial Differential Equations*, vol. I & II., Oxford Lecture Series in Mathematics and its Applications: Oxford University Press.
33. Virga, E.G. 1994. *Variational theories for liquid crystals*, vol. 8., Applied mathematics and mathematical computation London: Chapman & Hall.
34. White, B. 1988. Homotopy classes in Sobolev spaces and the existence of energy minimizing maps. *Acta Mathematica* 160: 1–17.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.